

# Moduli restriction and Chiral Matter in Heterotic String Compactifications

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## Abstract

Supersymmetric heterotic string models, built from a stable holomorphic vector bundle  $V$  on a Calabi-Yau threefold  $X$ , usually come with many vector bundle moduli whose stabilisation is a difficult and complex task. It is therefore of interest to look for bundle constructions which, from the outset, have as few as possible bundle moduli. One way to reach such a set-up is to start from a generic construction and to make discrete modifications of it which are available only over a subset of the bundle moduli space. Turning on such discrete 'twists' constrains the moduli to the corresponding subset of their moduli space: the twisted bundle has less parametric freedom. We give an example of a set-up where this idea can be considered concretely. Such non-generic twists lead also to new contributions of chiral matter (which greatly enhances the flexibility in model building); their computation constitutes the main issue of this note.

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# 1 Introduction

A supersymmetric heterotic string model in four dimensions (4D) is given by the low energy effective theory arising from a compactification of the ten-dimensional heterotic string on a Calabi-Yau threefold  $X$  endowed with a polystable holomorphic vector bundle  $V'$ . Often one takes  $V' = (V, V_{hid})$  with  $V$  a *stable* bundle embedded in the visible  $E_8$  whose commutant gives the unbroken gauge group in 4D ( $V_{hid}$  plays the same role for the hidden  $E_8$ ). We restrict our attention to  $V$  (and will assume  $c_1(V) = 0$ ).

Besides the Kahler and complex structure moduli of  $X$  one gets moduli from the parameters of the bundle construction. As for the other moduli one searches for mechanisms, like world-sheet instantons and the superpotential generated by them, to stabilise these moduli to particular values; at least one wants to restrict their freedom to certain subloci of the moduli space, thereby simplifying the problem. So it is of interest to have a bundle construction which, from the outset, comes with as few as possible bundle moduli.

One way to reach such a set-up is to start with a rather generic bundle construction and to make twists which are available only over a subset  $\mathcal{S}$  of the bundle moduli space  $\mathcal{M}_V$ : turning on such a twist will restrict the moduli to  $\mathcal{S}$  if the twist is discrete.

We describe in the following a set-up where this idea can be considered concretely. We emphasize from the outset that, although the moduli reducing effect of the new twists is the rationale which lies behind our motivation to consider them, we will focus in the present paper on another interesting and phenomenologically relevant effect of the new twists. Besides the issue of moduli stabilisation (or at least reduction) the other prominent issue is the influence of such twists on the cohomological invariants of  $V$ , specifically the net generation number  $N_{gen} = h^1(X, V) - h^1(X, V^*) = -\chi(V) = -\int ch(V)Td(X) = -\frac{1}{2}c_3(V)$  (considered as number) and the impact for the anomaly cancellation condition involving  $c_2(V)$  and  $c_2(X)$ . We will therefore compute in our concrete *set-up* the new contributions to  $c_3(V)$  and  $c_2(V)$  provided by the mentioned twists (and in the even more concrete *examples* of twists in this set-up which we will present we can evaluate the ensuing expressions even further).

Let us describe some related work. In the set-up we will choose, the case of spectral cover bundles on elliptically fibered  $X$ , [1] constitutes the basic reference. There the more general possibility of using 'non-standard' twists in the sense described was already seen (including, on a more implicit level, the corresponding issue of moduli restriction) but, at the early stage of the investigations in the field at that time, the detailed discussion of the standard twist was sufficient for all purposes. Quite generally the point in this

issue is to give worked out examples for non-generic twists (and making explicit the moduli reduction and the influence on the Chern classes) which was not in the focus of [1]. The issue reappeared at the surface on the occasion of investigations of dual  $F$ -theory models; in that language the question is discussed in [3] where also the general philosophy of using a non-standard twist (with ensuing moduli reduction) is exemplified by a specific construction leading to a three-generation model. The examples given in the present paper are different, not just for the  $SU(5)$  case and given directly in the heterotic set-up (though one can also use a 'heterotic language' directly in the  $F$ -theory set-up); furthermore computed is here not only the influence in the chiral matter expressed by the change in  $c_3(V)$  but also the change in  $c_2(V)$  (displaying also the specific parameter freedom in these Chern classes for the cases of our examples).

It is interesting to note that the issue of moduli reduction by using such special objects has not only be further explored in the  $F$ -theory context, for example in [5] (to quote just one paper from that direction of research); in a broader sense the issue converges also with another line of research in the heterotic context: in [4] heterotic constructions are made which exist only for a subset of the complex structure moduli, leading to a corresponding reduction of freedom in that moduli space.

### *Structure of the paper*

To clarify the development of our argument let us point to a hierarchy of set-ups which become more and more concrete. In *sect. 2* we describe a concrete set-up which constitutes the first and most general layer; there we make the general idea of twisting concrete by pointing to the 'new' discrete twists which are possible in the spectral cover scenario of bundle construction over an elliptically fibered space  $\pi : X \rightarrow B$ ; here the mentioned twists can be handled effectively: we compute their impact on the Chern classes. Then in a second, already more concrete layer we specialise to certain general classes of 'new' (i.e. non-generic) divisors on the spectral cover surface and compute their new cohomological contributions, thus making our previous general formulae explicit for these cases (whereas the moduli reduction effect in these examples is just suggested). Our first example (second layer) is in *sect. 3*; in *sect. 4* we give another example of the type of twist class one can use in this set-up; we also discuss the issue of moduli stabilisation (or rather restriction) in this connection. In a third and final layer of concreteness we give in *sect. 3.3.1* and *sect. 4.3* explicit examples of the types of twist class described for the two most common cases of  $B$ , the case of a Hirzebruch surface and of a del Pezzo surface, respectively, thereby giving concrete examples of the general type of classes described in the second layer and evaluating our formulae for them. We conclude in *sect. 5*.

## 2 A concrete set-up

Let us consider spectral  $SU(n)$  vector bundles on an elliptic Calabi-Yau space  $\pi : X \rightarrow B$  with section  $\sigma$ . (We will identify notationally  $\sigma$ , its image and the divisor and cohomology class of that image; we also use the notation  $c_1 := c_1(B)$ , often with the pull-back to  $X$  or  $C$  understood; one has  $\sigma^2 = -c_1\sigma$ , cf. [1].)

In this case one has

$$V = p_*(\mathcal{P} \otimes p_C^*L) \quad (2.1)$$

where one has the following objects (this construction is by now fairly standard, cf. [1]): one chooses a (ramified)  $n$ -fold cover surface  $C \subset X$  over  $B$ , of cohomology class  $n\sigma + \pi^*\eta$  with  $\eta \in H^{1,1}(B)$ , and a line bundle  $L$  over  $C$ ;  $\mathcal{P}$  is the Poincare bundle over  $X_{(1)} \times_B X_{(2)}$  restricted here to  $X \times_B C$  and  $p$  and  $p_C$  the projections to the first and second factor, respectively (here one has  $c_1(\mathcal{P}) = \Delta - \sigma_1 - \sigma_2 - c_1$  with the diagonal class  $\Delta$  in the fibre product and the corresponding section classes from the factors; all necessary pull-backs are understood).

The condition  $c_1(V) = 0$  will fix  $c_1(L)$  in  $H^{1,1}(C) \cap H^2(C, \mathbf{Z})$  up to a class  $\gamma$  in  $\ker(\pi_{C*})$ :

$$c_1(L) = \frac{n\sigma + \eta + c_1}{2} + \gamma \quad (2.2)$$

where one has  $\pi_{C*}\gamma = 0$  (here  $\pi_C : C \rightarrow B$  is the restricted projection; we will usually suppress the pull-back notation and write just  $\phi$  for  $\pi^*\phi$  or  $\pi_C^*\phi$ ).

The equation for  $C$  is given by

$$w = a_0z + a_2x + a_3y = 0 \quad (2.3)$$

$$w = a_0z^2 + a_2xz + a_3yz + a_4x^2 + a_5xy = 0 \quad (2.4)$$

for  $n = 3$  and  $n = 4$  or  $5$ , resp. (with  $a_5 = 0$  for  $n = 4$ ; here  $x, y, z$  are Weierstrass coordinates of the elliptic fibre and  $a_i$  sections of suitable line bundles over  $B$ ).

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<sup>2</sup>There are further conditions which have to be satisfied and have to be checked in detail in any concrete example (as we will do later). Note first that the effectiveness of  $C$  entails the effectiveness of  $\eta$ ; furthermore the irreducibility of  $C$  (which one needs to assume for the stability of  $V$ ) is given just for  $\eta - nc_1$  effective and the linear system  $|\eta|$  being base-point free; the latter condition is best investigated further explicitly on the different standard choices for the basis  $B$ : it holds on a Hirzebruch surface  $\mathbf{F}_k$  if  $\eta \cdot b \geq 0$  and on a del Pezzo surface  $\mathbf{dP}_k$  with  $2 \leq k \leq 7$  if  $\eta \cdot E \geq 0$  for all curves  $E$  with  $E^2 = -1$  and  $E \cdot c_1 = 1$  (such curves generate the effective cone) (for the notation used cf. sect. 4.3 where this information is used; similarly also in sect. 3.3.1). We remark further that one has also  $c_2(X) = 12c_1\sigma + 11c_1^2 + c_2(B)$  (cf. [1]).

If one assumes that  $C$  is ample one has  $H^{1,0}(C) = 0$  and  $L$  is determined by its first Chern class (no further continuous moduli occur); then also the curve  $A_B := C \cap B \subset B$  is ample ( $A_B$ , or  $A_C$  later, will also denote the cohomology class). We will, however, have reason to consider also the case that this curve, and thus  $C$  as well, is not ample (cf. sect. 4.3). In this case further, continuous degrees of freedom, related to  $H^1(C, \mathcal{O})/H^1(C, \mathbf{Z})$ , occur which are fibered over the discrete classification of the line bundles provided by the Chern class. We nevertheless continue to speak of *discrete* twists also then, as the important point for us is that (regardless of the additional continuous degree of freedom in the fibre of this situation) the use of a 'new' twist (not belonging to the standard twists available generically) can not be turned off continuously, i.e. the effect of reduction in the vector bundle moduli space, in which we are interested, takes place in any case.

## 2.1 The standard situation

If one wants to describe the possible freedom one has in choosing  $\gamma$ , one can say generically only the following: the only obvious classes on  $C$  are, besides the section  $\sigma|_C$ , the pull-back classes  $\pi^*\phi$  where the class  $\phi$  comes from the base. One finds [1] that  $\pi_{C*}\sigma|_C = \bar{\eta} := \eta - nc_1$  and so the only class in  $\ker(\pi_{C*})$  available in general is

$$\gamma = n\sigma|_C - \pi_C^*\bar{\eta} \quad (2.5)$$

(or suitable multiples  $\lambda\gamma$  of it; at this point an integrality issue occurs<sup>3</sup> which we do not need to make explicit here; important is that  $\lambda$  has only discrete freedom).

One gets the following formulae (cf. [1] and [2]; for  $p : X \times_B C \rightarrow X$  cf. above)

$$c_2(V) = \eta\sigma - \frac{n^3 - n}{24}c_1^2 - \frac{n}{8}\eta\bar{\eta} - \frac{1}{2}\pi_{C*}\gamma^2 \quad (2.6)$$

$$\frac{1}{2}c_3(V) = \frac{1}{2}p_*(\gamma c_1^2(\mathcal{P})) \quad (2.7)$$

In this form the formulae hold for a *general*  $\gamma$ . With the *concrete generic*  $\gamma$  class given above one finds<sup>4,5</sup>

$$\pi_{C*}\gamma^2 = -n\eta\bar{\eta} \quad (2.8)$$

$$\frac{1}{2}p_*(\gamma c_1^2(\mathcal{P})) = \eta\bar{\eta} \quad (2.9)$$

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<sup>3</sup> $\lambda$  has to be half-integral in a specific way depending on the parity of  $n$  ( $n$  odd needs  $\lambda \in \frac{1}{2} + \mathbf{Z}$  while  $n$  even needs  $\eta \equiv c_1(2)$  for  $\lambda \in \mathbf{Z}$  and  $0 \equiv c_1(2)$  for  $\lambda \in \frac{1}{2} + \mathbf{Z}$ )

<sup>4</sup>this corresponds to the choice  $\lambda = 1$ ; for  $n$  odd  $\lambda$  has to be strictly halfintegral, but it is obvious how the expressions have to be adapted: (2.8) and (2.9) come with a factor  $\lambda^2$  and  $\lambda$ , resp., in general

<sup>5</sup>the final term on the right hand side of (2.9) actually occurs at first as  $\sigma\eta\bar{\eta}$ , i.e.  $\sigma\pi^*\eta\pi^*\bar{\eta}$ ; as this is interpreted in any case as a number one can simply read it as intersection number on  $B$

## 2.2 The new, extended class of twists

Now let us assume that, at least for a certain subset  $\mathcal{S}$  of the moduli space  $\mathcal{M}_V$ , further divisor classes on  $C$  exist (such that further corresponding cohomology classes, denoted by  $\tilde{\chi}$  below, in the expression for  $\gamma$  can occur). Then we can make a more general ansatz for the cohomology class  $\gamma$  (where  $\rho$  here is still a class coming from the base)<sup>6</sup>

$$\gamma = n\sigma + \rho + \tilde{\chi} \quad (2.10)$$

The condition  $\pi_{C*}\gamma = 0$  amounts now to  $n(\bar{\eta} + \rho) + \pi_{C*}\tilde{\chi} = 0$ ; to secure the divisibility of  $\pi_{C*}\tilde{\chi}$  by  $n$  we are led to the slightly modified ansatz  $\tilde{\chi} := n\chi$ , that is

$$\gamma = n(\chi + \sigma) + \rho = n(\chi + \sigma) - \pi_{C*}(\chi + \sigma) \quad (2.11)$$

In the last rewriting we made manifest the condition on  $\rho$  which guarantees<sup>7</sup>  $\gamma \in \ker(\pi_{C*})$  (in the final expression one can also turn off, discretely,  $\chi$  to get back (2.5)). Again one may also consider suitable multiples  $\lambda\gamma$  and an integrality issue occurs<sup>8</sup>.

Now we are interested in the new contributions to the Chern classes arising from the *new* class  $\chi$ , i.e., from the class which is *not* already contained in the span of the classes which are generically present (which consist, besides the special class  $\sigma$  (i.e.  $\sigma|_C$ ), in the pull-back classes  $\pi_C^*\phi$ ). It is useful to recall in this context the 'projection formula'  $\pi_{C*}(\pi_C^*\phi \cdot \sigma) = \phi \cdot \pi_{C*}\sigma$  involving pull-back classes. As one has  $\pi_{C*}\pi_C^*\phi = n\phi$  one can write then also  $n \pi_{C*}(\pi_C^*\phi \cdot \sigma) = \pi_{C*}\pi_C^*\phi \cdot \pi_{C*}\sigma$ . Therefore, from classes  $\chi$  (which are *not* pull-back classes like  $\pi_C^*\phi$ ) one can expect, as new contributions, non-zero terms built from a corresponding difference of the right and the left hand side of this relation, i.e. terms like  $\pi_{C*}\chi \cdot \pi_{C*}\sigma - n \pi_{C*}(\chi \cdot \sigma)$ , or, more generally,  $\pi_{C*}\chi \cdot \pi_{C*}\zeta - n \pi_{C*}(\chi \cdot \zeta)$  (where the further class  $\zeta$  on  $C$  could be  $\chi$  itself, for example, cf. (2.13) below).

One gets now indeed

$$\pi_{C*}\gamma^2 = -n \left[ \left( \pi_{C*}(\chi + \sigma) \right)^2 - n \pi_{C*}(\chi + \sigma)^2 \right] \quad (2.12)$$

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<sup>6</sup>to avoid unclear notation we now denote the pull-back class by  $\rho$  instead of the former  $-\bar{\eta}$  (cf. (2.5)); the pull-back operation itself is suppressed, so  $\rho$  is actually  $\pi_C^*\rho$ ; if no confusion can arise we will also suppress in the following the restriction and write just  $\sigma$  for the class  $\sigma|_C$

<sup>7</sup>To avoid any confusion note that the final term  $\pi_{C*}(\chi + \sigma)$  is, in itself, a class projected down to  $B$ ; if it occurs, as it is the case here, in a formula for a class on  $C$  (the class  $\gamma$ ), then this means that it has to be read as being pulled-back to  $C$ ; in other words this means actually the class  $Q := \pi_C^*\pi_{C*}(\chi + \sigma)$  (for  $\chi = 0$  one gets back  $Q = \pi_C^*\bar{\eta}$ , cf. (2.5)); so both terms in the final expression  $P - Q$  on the right hand side of (2.11) fulfil  $\pi_{C*}P = n\pi_{C*}(\chi + \sigma) = \pi_{C*}Q$ , thus giving indeed  $\gamma \in \ker(\pi_{C*})$ .

<sup>8</sup> $\lambda$  has to be half-integral in a specific way depending on the parity of  $n$  ( $n$  odd needs  $\lambda \in \frac{1}{2} + \mathbf{Z}$  and  $\chi \equiv 0 \pmod{2}$  while  $n$  even needs  $\eta \equiv c_1 \pmod{2}$  for  $\lambda \in \mathbf{Z}$  and  $\pi_{C*}\chi \equiv c_1 \pmod{2}$  for  $\lambda \in \frac{1}{2} + \mathbf{Z}$ )

$$\begin{aligned}
= & -n \left[ \pi_{C*} \sigma \cdot \pi_{C*} \sigma - n \pi_{C*} \sigma^2 \right. \\
& + 2 \left( \pi_{C*} \chi \cdot \pi_{C*} \sigma - n \pi_{C*} (\chi \cdot \sigma) \right) \\
& \left. + \pi_{C*} \chi \cdot \pi_{C*} \chi - n \pi_{C*} \chi^2 \right] \quad (2.13)
\end{aligned}$$

Note that here the first line in the big brackets on the right hand side in (2.13) is the usual term  $\bar{\eta}\bar{\eta} + nc_1\bar{\eta} = \eta\bar{\eta}$ , cf. (2.8). The additional, new contributions in the last two lines are now indeed of the expected form for which we argued in the previous paragraph.

And similarly one gets (using  $\sigma_i \cdot c_1(\mathcal{P}) = 0$ )

$$c_3(V) = p_* \left( \gamma c_1^2(\mathcal{P}) \right) = p_* \left( (\rho + n\chi) \left( -2\sigma_2\sigma_1 + c_1(-3\Delta + \sigma_1 + \sigma_2) \right) \right) \quad (2.14)$$

$$= \left( -2\rho(\bar{\eta} + nc_1)\sigma_1 - 2n\pi_{C*}(\chi\sigma_2 + \chi c_1)\sigma_1 \right) \quad (2.15)$$

This leads, after using  $\rho = -\bar{\eta} - \pi_{C*}\chi$ , to the further evaluation

$$-N_{gen} = -\rho\eta - n\pi_{C*}(\chi(\sigma + c_1)) \quad (2.16)$$

$$= \eta\bar{\eta} + \eta\pi_{C*}\chi - n\pi_{C*}(\chi(\sigma + c_1)) \quad (2.17)$$

$$= \eta\bar{\eta} + \bar{\eta}\pi_{C*}\chi - n\pi_{C*}(\chi\sigma) \quad (2.18)$$

This gives the final formula for the generation number which shows that the new contribution is just of the structurally expected type (cf. (2.9))

$$-N_{gen} = \eta\bar{\eta} + \pi_{C*}\chi \cdot \pi_{C*}\sigma - n\pi_{C*}(\chi \cdot \sigma) \quad (2.19)$$

So let us finally list (using again  $\pi_{C*}\sigma = \bar{\eta} = \eta - nc_1$ ) the complete expressions one gets if one turns on, as specified in (2.11), a non-pull-back class  $\tilde{\chi} = n\chi$  in the twist

$$\begin{aligned}
c_2(V) = & \eta\sigma - \frac{n^3 - n}{24}c_1^2 - \frac{n}{8}\eta\bar{\eta} \\
& + n\lambda^2 \left[ \frac{1}{2}\eta\bar{\eta} + \pi_{C*}\chi \cdot \pi_{C*}\sigma - n\pi_{C*}(\chi \cdot \sigma) + \frac{1}{2}(\pi_{C*}\chi \cdot \pi_{C*}\chi - n\pi_{C*}\chi^2) \right] \quad (2.20)
\end{aligned}$$

$$-N_{gen} = \lambda \left[ \eta\bar{\eta} + \pi_{C*}\chi \cdot \pi_{C*}\sigma - n\pi_{C*}(\chi \cdot \sigma) \right] \quad (2.21)$$

(the first terms in the [...] brackets are the standard terms, the rest the corrections).

In the remaining sections we want to give examples of this construction, i.e. concrete classes to twist with and the corresponding evaluation of the new cohomological contributions; furthermore we want to make remarks on the issue of moduli reduction. But before we come to this let us consider two related issues: the direct chiral matter computation of  $N_{gen}$  and the set of classes which are available in general for  $\chi$ .

## 2.3 The direct chiral matter computation of $N_{gen}$

Let us first recall (cf. [2]) the computation of  $N_{gen}$  from the net amount  $h^1(X, V) - h^1(X, V^*)$  of chiral matter for the standard  $\gamma$  twist. This proceeds, as  $H^1(X, V)$  is localised along  $\pi^*A_B$  and by noting that  $V|_B \cong \pi_{C*}L$ , with the help of the Leray spectral sequence (which itself simplifies because of  $R^0\pi_*V = 0$ )

$$0 \longrightarrow H^1(B, R^0\pi_*V) \longrightarrow H^1(X, V) \longrightarrow H^0(B, R^1\pi_*V) \rightarrow H^2(B, R^0\pi_*V) \quad (2.22)$$

One computes (taking into account the relative Serre duality  $(R^1\pi_*V)^* \cong \pi_*(V^* \otimes K_B^*)$ )

$$H^1(X, V) \cong H^0(B, R^1\pi_*V) \cong H^0(A_B, R^1\pi_*V|_{A_B}) \quad (2.23)$$

$$\cong H^0\left(A_B, \left[L|_{A_C} \otimes \pi_C^*K_B|_{A_C}\right]_{A_B}\right) \quad (2.24)$$

where the internal brackets in the final expression indicate that the line bundle inside them, which a priori lives on  $A_C := \sigma|_C$ , is interpreted on  $A_B$ . One gets for  $N_{gen}$  the result (we put again  $\lambda = 1$ ; the brackets with subscript  $B$  indicate that the intersection product on  $C$  inside them is interpreted afterwards as an intersection product on  $B$ )

$$\chi\left(A_B, \left[L|_{A_C} \otimes \pi_C^*K_B|_{A_C}\right]_{A_B}\right) = -\frac{1}{2} \deg K_{A_B} + \deg[L|_{A_C}]_{A_B} + \deg K_B|_{A_B} = [\gamma \cdot A_C]_B \quad (2.25)$$

$$= [\gamma \cdot \sigma|_C]_B = [-\eta \cdot \sigma|_C]_B = -\pi_{C*}(\eta \cdot \sigma|_C) \quad (2.26)$$

$$= -\eta\bar{\eta} \quad (2.27)$$

where we have inserted the relation (we have also used  $\deg K_A = \deg K_C|_A + \deg K_B|_A$ )

$$\deg[L|_{A_C}]_{A_B} = \deg L|_{A_C} = \frac{1}{2}(\deg K_C - \deg K_B)|_{A_C} + \gamma \cdot A_C \quad (2.28)$$

$$= \frac{1}{2} \deg K_A - \deg K_B|_A + \gamma \cdot A_C \quad (2.29)$$

(if the curve  $A$  does not carry a subscript, indicating in which surface,  $C$  or  $B$ , is has to be interpreted, then that does not matter).

Now in the new, more general case one gets

$$\begin{aligned} \gamma \cdot \sigma &= n\chi \cdot \sigma - nc_1\sigma - (\pi_C^*\pi_{C*}\chi)\sigma - \bar{\eta}\sigma \\ &= n\chi \cdot \sigma - (\pi_C^*\pi_{C*}\chi)\sigma - \eta\sigma \end{aligned} \quad (2.30)$$

The latter expression projects under  $\pi_{C*}$  down to

$$\pi_{C*}(\gamma \cdot \sigma) = n\pi_{C*}(\chi \cdot \sigma) - \pi_{C*}\chi \cdot \pi_{C*}\sigma - \eta\bar{\eta} \quad (2.31)$$

Thus, for  $-N_{gen}$ , we arrive again at the expression (2.19).



## 2.4 Remarks on the classes available for $\chi$

Let us quantify the available resources for classes like  $\chi$ . From the outset one has just the class  $\sigma|_C$  and the pull-back classes  $\pi_C^*\phi$  at one's disposal; so the number of classes which are available generically is

$$1 + h^{1,1}(B) = e(B) - 1 \quad (2.32)$$

(if one makes furthermore use of the fact that  $B$  is a rational surface of  $1 = p_g(B) = 1 - h^{1,0}(B) + h^{2,0}(B) = (c_1^2 + e(B))/12$ , using Noether's formula, one obtains here the alternative evaluation  $11 - c_1^2$ ).

On the other hand the number of classes available in principle is given by the rank of the lattice  $H^2(C, \mathbf{Z}) \cap H^{1,1}(C)$ ; here one computes<sup>9</sup> for  $H^{1,1}(C)$  itself

$$h^{1,1}(C) = 2n\eta\bar{\eta} + \frac{4n^3 - 5n}{6}c_1^2 + 10\eta c_1 + \frac{5n}{6}e(B) + 2h^{1,0}(C) \quad (2.33)$$

(if one makes again use of the fact that  $B$  is a rational surface one obtains here the alternative evaluation  $2n\eta\bar{\eta} + \frac{4n^3 - 10n}{6}c_1^2 + 10\eta c_1 + 10n + 2h^{1,0}(C)$ ). It now depends on the complex structure of  $C$  which of these forms have integral periods when integrated against a basis of integral cycles, and thus belong also to  $H^2(C, \mathbf{Z})$ . In our situation the actually available complex structures of  $C$  come from its 'motions' in the ambient space  $X$ , i.e. from the possible different equations (up to an overall rescaling) for  $C$  in  $X$ ; this, as described, comprises just the continuous part of the moduli space  $\mathcal{M}_V$  of the bundle.

One can view the problem to determine the intersection  $H^2(C, \mathbf{Z}) \cap H^{1,1}(C)$  also from the other side: the demand that a topological class  $\xi \in H^2(C, \mathbf{Z})$  has type  $(1, 1)$  (so is related to a holomorphic cycle) is expressed by the orthogonality  $\xi \perp \alpha$  for all  $\alpha$  in a base of the subspace which constitutes  $H^{2,0}(C)$ . A priori each  $\alpha$  could be everywhere in  $U = \{\beta \in H^2(C, \mathbf{R}) | \beta \wedge \beta = 0, \beta \wedge \bar{\beta} > 0\}$ . So classes  $\xi$  are relatively scarce (cf. [1], sect. 7.4).

A well-known analogous situation is that of a  $K3$  surface where the demand for a higher rank (the Picard number) of the span of the sought-after classes  $\xi$  restricts one accordingly in the moduli space.<sup>10</sup>

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<sup>9</sup>using  $c_1(C) = -(n\sigma + \eta)|_C$  and  $c_2(C) = C^2|_C + c_2(X)|_C$ , cf. [1], which give  $c_1^2(C) = 3n\eta\bar{\eta} + n^3c_1^2$  and  $e(C) = 3n\eta\bar{\eta} + (n^3 - n)c_1^2 + 12\eta c_1 + ne(B)$ , from which one derives in turn, using Noether's formula now applied to  $C$ , that  $h^{2,0}(C) - h^{1,0}(C) = \frac{n}{2}\eta\bar{\eta} + \frac{n^3 - n/2}{6}c_1^2 + \eta c_1 + \frac{n}{12}e(B) - 1$

<sup>10</sup>But note that in that example the relevant moduli space are the possible positions of  $H^{2,0}(K3)$  in  $\mathbf{P}(U)$ ; by contrast in our case those moduli of  $C$  which are relevant for the spectral bundle set-up are not exactly the internal complex structures of  $C$  but the external motions in  $X$ , i.e.  $H^{2,0}(C)$  itself.

### 3 A first example of a non-generic twist class

In the two examples of a non-generic twist class given in the present and in the next chapter we will use the idea that under special conditions on the (bundle) moduli one of the generically present classes  $\pi_C^* \phi$  and  $\sigma|_C$  becomes reducible; then a component of this reducible class represents a 'new' class to twist with.

In this section we take the first case: we will look for a case where under certain conditions the preimage  $\mathcal{C} := \pi_C^{-1}(c)$  of a curve  $c \subset B$  becomes reducible in  $C$

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 \quad (3.1)$$

We will assume  $n > 3$  and to be as concrete as possible we choose the cases  $n = 4$  or  $n = 5$  (which are also phenomenologically the most important ones; the factorisation idea described in the following can be analogously considered for  $n > 5$ ). The spectral cover equation is (with  $a_5 = 0$  for  $n = 4$ ; here  $a_i$  are global sections of<sup>11</sup>  $\mathcal{O}_B(\eta - ic_1)$ )

$$w = a_0 z^2 + a_2 xz + a_3 yz + a_4 x^2 + a_5 xy \quad (3.2)$$

Now let us consider in a first, *preliminary* step the following factorisation

$$w = (f_1 z + g_1 x + h_1 y)(f_2 z + g_2 x) \quad (3.3)$$

(with  $h_1 = 0$  for  $n = 4$ ) where  $f_1, g_1, h_1, f_2, g_2$  are sections of suitable line bundles over  $B$ : if one denotes the vanishing divisor of  $g_2$ , say, by  $(g_2)$  one has

$$f_1 \in H^0(B, \mathcal{O}_B(\eta - 2c_1 - (g_2))) \quad (3.4)$$

$$g_1 \in H^0(B, \mathcal{O}_B(\eta - 4c_1 - (g_2))) \quad (3.5)$$

$$h_1 \in H^0(B, \mathcal{O}_B(\eta - 5c_1 - (g_2))) \quad (3.6)$$

$$f_2 \in H^0(B, \mathcal{O}_B(2c_1 + (g_2))) \quad (3.7)$$

$$g_2 \in H^0(B, \mathcal{O}_B((g_2))) \quad (3.8)$$

The relations to the original coefficients are

$$a_0 = f_1 f_2 \quad (3.9)$$

$$a_2 = f_1 g_2 + g_1 f_2 \quad (3.10)$$

$$a_3 = h_1 f_2 \quad (3.11)$$

$$a_4 = g_1 g_2 \quad (3.12)$$

$$a_5 = h_1 g_2 \quad (3.13)$$

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<sup>11</sup>with a common abuse of notation to denote divisors by symbols for corresponding cohomology classes

If the original coefficients  $a_i$  can be written in this rather special way one gets the relation

$$a_0 a_5^2 - a_2 a_3 a_5 + a_3^2 a_4 = 0 \quad (3.14)$$

( $a_3=0$  for  $n=4$ ). Note that this means here identical vanishing over all of  $B$ . If considered as an equation for a curve in  $B$  it describes [1] the localization curve of the bundle  $\Lambda^2 V$ .

We have not yet considered the question whether the relation (3.14), which as we showed is necessary to have a factorization like (3.3), is also sufficient to have such a relation. We will consider the question further in a moment in the somewhat reduced framework of factorization in which we are actually interested and to which we turn now.

The factorizability considered above is much more than we actually have to demand. Let  $c$  denote a (smooth irreducible reduced) curve in  $B$  and assume the following factorisability of  $w$  over the elliptic surface  $\mathcal{E}_c := \pi^{-1}(c)$  (with  $F_1 := f_1|_c$  and so on)

$$w|_{\mathcal{E}_c} = (F_1 z + G_1 x + H_1 y)(F_2 z + G_2 x)|_{\mathcal{E}_c} \quad (3.15)$$

where  $F_1, G_1, H_1, F_2, G_2$  are now sections of suitable line bundles over  $c$ : for example one has that  $F_1 \in H^0(c, \mathcal{O}_c((\eta - 2c_1)|_c - (G_2)))$  and  $A_0 = F_1 F_2$  (with  $A_i := a_i|_c$ ) and analogous expressions for all the other equations.

So one gets now as condition for the factorizability over  $c$  that the curve given in<sup>12</sup> (3.14) has  $c$  as a component, i.e. that the equation (3.14) is fulfilled along  $c$  (the concrete case in which we are interested is  $c \cong \mathbf{P}^1$  which we assume now for simplicity)

$$A_0 A_5^2 - A_2 A_3 A_5 + A_3^2 A_4 = 0 \quad (3.16)$$

Note that the relation just presented is not only necessary but also sufficient to have (3.15) (we assume here  $n = 5$ ). Note first that because of (3.16) one has  $A_5|A_3 A_4$ , so that one can write  $A_5 = H_1 G_2$  with  $H_1|A_3$  and  $G_2|A_4$ ; let us write furthermore  $A_4 = G_1 G_2$  and  $A_3 = H_1 F_2$ . From (3.16) one gets  $A_5 F_2 = A_3 G_2|A_0 A_5$ , such that  $F_2|A_0$  and one can write  $A_0 = F_1 F_2$ . From these determinations it follows already, once more with (3.16), that  $A_2 = (A_0 A_5^2 + A_3^2 A_4)/A_3 A_5 = F_1 G_2 + G_1 F_2$ .

If condition (3.15) is fulfilled one has the decomposition (3.1) with  $\mathcal{C}_1$  and  $\mathcal{C}_2$  corresponding to the first and second factors in (3.15), respectively: in other words the five-fold cover  $\mathcal{C}$  of  $c$  decomposes into a triple cover  $\mathcal{C}_1$  and a double cover  $\mathcal{C}_2$  (this is for  $n = 5$ ; for  $n = 4$  one has to adjust these assertions, cf. sect. 3.3).

For future reference we note the relation<sup>13</sup> (until sect. 3.3 we assume now  $n = 5$ )

$$A_B = \bar{\eta} = (a_5) = (h_1) + (g_2) \quad (3.17)$$

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<sup>12</sup>which is now read as an equation for a curve in  $B$  and not as an identical vanishing over all of  $B$

<sup>13</sup>where the divisor  $A_B$  and the zero divisors  $(a_5), (h_1)$  and  $(g_2)$  denote also the cohomology classes

To compute the contributions in (2.20) and (2.21) for  $\chi = \mathcal{C}_2$ , say, it remains, having  $\pi_{C*}\mathcal{C}_2 = 2c$ , to compute  $\pi_{C*}(\mathcal{C}_2 \cdot \sigma|_C)$  and  $\pi_{C*}(\mathcal{C}_2^2)$ . For this note first that  $\mathcal{C}_1 \cdot \sigma|_C + \mathcal{C}_2 \cdot \sigma|_C = \mathcal{C} \cdot \sigma|_C = \mathcal{E}_c|_C \cdot \sigma|_C = \mathcal{E}_c \cdot \sigma \cdot C = \mathcal{E}_c|_\sigma \cdot C|_\sigma = c \cdot A_B = (h_1) \cdot c + (g_2) \cdot c = \deg H_1 + \deg G_2$  because of (3.17); in particular one has  $\mathcal{C}_2 \cdot \sigma|_C = (g_2) \cdot c$ .

To compute  $\pi_{C*}(\mathcal{C}_2^2)$  let us compute first  $\mathcal{C}_1 \cdot \mathcal{C}_2$  (intersection number in  $C$ ). For this note the following determinations of the cohomology classes of involved divisors<sup>14</sup>

$$\mathcal{C}_1 = (f_1 z + g_1 x + h_1 y)|_{\mathcal{E}_c} = (3\sigma + \eta - 2c_1 - (g_2))|_{\mathcal{E}_c} \quad (3.18)$$

$$\mathcal{C}_2 + \sigma|_{\mathcal{E}_c} = (f_2 z + g_2 x)|_{\mathcal{E}_c} = (3\sigma + 2c_1 + (g_2))|_{\mathcal{E}_c} \quad (3.19)$$

(divisors in the surface  $\mathcal{E}_c$ ). Thus one gets (as intersection number in  $\mathcal{E}_c$  and also<sup>15</sup> in  $C$ )

$$\mathcal{C}_1 \cdot \mathcal{C}_2 = (2\bar{\eta} + 6c_1 + (g_2))c \quad (3.20)$$

Additional standard assumptions which are often adopted (though not strictly necessary) are that  $\bar{\eta}$ , which is effective, is even ample. If one assumes furthermore  $c_1$  effective one excludes from the standard examples<sup>16</sup> for  $B$  only the Enriques surface; and if one assumes  $c_1$  even to be ample one excludes only  $\mathbf{F}_2$  in addition. Making these assumptions the first two terms on the right hand side of (3.20) are  $> 0$ ; so for  $\mathcal{C}_1 \cdot \mathcal{C}_2 = 0$  one then would need  $(g_2)c < 0$ , in particular neither of the effective divisors  $(g_2)$  and  $c$  could be ample.

With this information and the projection formula  $\pi_{C*}(\mathcal{C}_2 \cdot \pi_C^* c) = \pi_{C*}(\mathcal{C}_2) \cdot c$  we get finally (note that for  $\pi_{C*}(\mathcal{C}_1^2)$  one gets  $3c$  instead of  $2c$  as first term in the final bracket)

$$\pi_{C*}(\mathcal{C}_2^2) = \pi_{C*}(\mathcal{C}_2 \cdot \mathcal{C} - \mathcal{C}_2 \cdot \mathcal{C}_1) = (2c - 2\bar{\eta} - 6c_1 - (g_2))c \quad (3.21)$$

Thus one gets finally from (2.20), (2.21) in this example of  $\chi = \mathcal{C}_2$  (as cohomology class) the complete expressions (the first terms in the [...] brackets on the right hand sides are the standard contributions, the terms proportional to  $c$  are the new contributions)

$$c_2(V) = \eta\sigma - 5c_1^2 - \frac{5}{8}\eta\bar{\eta} + 5\lambda^2 \left[ \frac{1}{2}\eta\bar{\eta} + \left( 2\bar{\eta} - 5(g_2) - 3c + 5(\bar{\eta} + 3c_1 + \frac{1}{2}(g_2)) \right) c \right] \quad (3.22)$$

$$-N_{gen} = \lambda \left[ \eta\bar{\eta} + (2\bar{\eta} - 5(g_2))c \right] \quad (3.23)$$

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<sup>14</sup>In (3.19) the equation in each fibre plane  $\mathbf{P}_{\mathbf{x},\mathbf{y},\mathbf{z}}^2$  is linear in the Weierstrass coordinates, so intersects the elliptic cubic *three* times; only two of these fibre points carry information (the fibre points  $q_1, q_2$  of  $\mathcal{C}_2$ ), a third one lies always at the zero point  $p_0$ : for  $f_2 z + g_2 x = z(f_2 + g_2 \frac{x}{z})$  shows as divisor three zeroes at  $p_0$  from  $z$  and two zeroes at  $q_1, q_2$  and a double pole at  $p_0$  from the affine part; by contrast for  $\mathcal{C}_1$  a *triple* pole cancels the zeroes of  $z$  while the affine part has three relevant zeroes (the fibre points of  $\mathcal{C}_1$ ).

<sup>15</sup>assuming that  $\mathcal{C}_1$  and  $\mathcal{C}_2$  have no common component such that no self-intersection number is involved

<sup>16</sup>Hirzebruch surfaces  $\mathbf{F}_k$  ( $k = 0, 1, 2$ ), del Pezzo surfaces  $\mathbf{dP}_k$  ( $k = 0, \dots, 8$ ) and the Enriques surface

Taking  $\chi = \mathcal{C}_1$  instead of  $\mathcal{C}_2$  gives, with  $(3\bar{\eta} - 5(h_1))c$  as new term in (3.23), by (3.17) the negative of the present new term (the same term  $(3\bar{\eta} - 5(h_1))c$  just replaces for  $\chi = \mathcal{C}_1$  the term  $(2\bar{\eta} - 5(g_2))c$  for  $\chi = \mathcal{C}_2$  in (3.22)).

One also has to take into account the parity considerations (cf. footn. 8). Here, in our case of  $n = 5$ , one has to check whether  $\chi = \mathcal{C}_1$  or  $\mathcal{C}_2$  is even when considered in the surface  $C$ . Now note first that the curve  $\mathcal{C}$ , in whose components  $\mathcal{C}_1$  and  $\mathcal{C}_2$  we are interested, can be considered as a curve either in the spectral cover surface  $C$  or in the elliptic surface  $\mathcal{E}_c = \pi^{-1}(c)$ : the representation as a divisor in these cases reads  $\mathcal{C} = \mathcal{E}_c|_C$  and  $\mathcal{C} = C|_{\mathcal{E}_c}$ , respectively. What one finds immediately from (3.18) and (3.19) is that *considered on the surface  $\mathcal{E}_c$*  only the class  $\chi = \mathcal{C}_2$  can be seen to be even and actually is so for  $\deg G_2$  even. This is, however, not related directly to the issue of being even on  $C$ . A *necessary* condition at least for the latter fact is that the curve class in question is even when *considered in the threefold  $X$* . Here one finds from  $\mathcal{C}_1 = (3\sigma + \eta - 2c_1 - (g_2)) \cdot \pi^{-1}(c)$  and  $\mathcal{C}_2 = (2\sigma + 2c_1 + (g_2)) \cdot \pi^{-1}(c)$  of curve classes in  $X$  (where  $\pi$  is the projection from  $X$  to  $B$  whereas  $\pi_C$  is the projection from  $C$  to  $B$ ) that this certainly holds if the class  $c$  in  $B$  is even or, in the case of  $\mathcal{C}_2$ , if  $(g_2) \cdot c = \deg G_2$  is even. But none of these conditions gives a sufficient condition for evenness on  $C$  of the curve class in question. The situation will be better in the case of  $n = 4$  considered below in sect. 3.3.

Another issue is whether one has to demand that the components  $\mathcal{C}_1$  and  $\mathcal{C}_2$  of  $\mathcal{C}$  do not intersect to make sure the smoothness of  $C$  (this is just to be on the save side; the spectral cover construction may make sense also in more general cases). The decomposition

$$C \cap \mathcal{E}_c = \mathcal{C}_1 + \mathcal{C}_2 \tag{3.24}$$

leads one to expect the picture that  $C$  decomposes *near  $\mathcal{E}_c$*  in two *local* branches given by a triple and a double cover (globally  $C$  will of course generically be irreducible). Potential intersection points of the two local branches do not necessarily have to be interpreted as a curve of double points of  $C$  as one does expect in any case ramification points of the covering  $\pi_C : C \rightarrow B$ . Despite the fact that double points are also possible to occur in principle, this generic presence of ramification points leads us here, in contrast to a similar case<sup>17</sup> in sect. 4, to adopt the strategy not to demand in addition that  $\mathcal{C}_1 \cdot \mathcal{C}_2 = 0$ .

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<sup>17</sup>We remark that in the sect. 4 where we investigate a similar example for 'new' classes on  $C$ , arising from components of the curve  $\sigma|_C$  which becomes reducible for special values of the moduli, the situation is at first somewhat similar: in both cases the question whether a reducibility of the intersection of  $C$  with a surface (here  $\mathcal{E}_c$ , there  $\sigma$ ) is dangerous for the smoothness of  $C$  is considered. Although there again in principle a harmless interpretation of the potential intersections is possible in analogy with what we have here, the expectation that these points are 'ramification-like' is much less standard there; so we will adopt the (highly-restrictive, as it turns out) condition  $\mathcal{D} \cdot \mathcal{D}' = 0$  in that latter case.

### 3.1 Why the component $\mathcal{C}_1$ of $\mathcal{C}$ represents a 'new' class

Let us now investigate whether the class (of the curve)  $\mathcal{C}_1$  on  $C$ , which according to its definition at least looks different from the generically available classes  $\sigma|_C$  and  $\pi_C^*\phi$ , is actually 'new', i.e. not contained in the span of these 'standard' classes.

For this let us assume that one would have a relation in cohomology (where  $k \in \mathbf{Z}$ )

$$\mathcal{C}_1 = k \sigma|_C + \pi_C^*\phi \quad (3.25)$$

The ensuing relation in  $H^2(B, \mathbf{Z})$ , which results from the projection  $\pi_{C*}$ , would then be

$$3c = kA_B + 5\phi \quad (3.26)$$

The class  $3c - k\bar{\eta}$ , however, will not in general<sup>18</sup> (the precise conditions have to be considered case by case) be divisible by 5 (for any  $k$ , assuming that not the class of  $c$  itself is already divisible by 5), giving the sought-after contradiction (similarly for  $n = 4$ ).

### 3.2 The question of moduli reduction

So if one restricts the bundle moduli (the degrees of freedom coming from the  $a_i$ ) by posing along  $c \cong \mathbf{P}^1$  the condition (3.16), one gets the factorization of the equation (3.15) for  $\mathcal{C}$  and thus the decomposition (3.1) which defines the 'new' cohomology class of  $\mathcal{C}_1$ . Asking conversely which moduli restriction is enforced by demanding the existence of this cohomology class (because it is used in a discrete twist) one meets the following problem: first what one really uses in the twist construction is a line bundle, thus a divisor *class* on  $C$ ; so one has to make sure that an *effective* representative in this class exists; in a second step one has to clarify whether the existence of such a curve (which we hope to play the role of  $\mathcal{C}_1$ ) can arise *only* in the way (3.1) or whether it may exist 'accidentally' already on a larger moduli subspace than the one given by (3.16) (where it exists 'naturally').

Let us consider the question on the numbers of degrees of freedom in the general versus the factorised case. To keep things simple in this illustrating example we did assume that  $c \cong \mathbf{P}^1$ . Then one gets as number of parameters of the general equation  $w|_{\mathcal{E}_c} = 0$  the sum of parameters in the homogeneous polynomials  $A_i$  of degree  $e - ir$  (where  $e := \eta \cdot c$  and  $r := c_1 \cdot c$  and we also assume here that  $e, r \geq 0$ ), so one gets in total  $5e - 14r + 5 - 1$ . On the other hand we have in the factorised case the degrees  $\deg F_1 = e - 2r - E$ ,  $\deg G_1 = e - 4r - E$ ,  $\deg H_1 = e - 5r - E$ ,  $\deg F_2 = 2r + E$ ,  $\deg G_2 =$   $E$  (with  $0 \leq E \leq e - 5r$  because of  $A_5 = H_1 G_2$ ), so  $3e - 9r - E + 5 - 1$  parameters in total.

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<sup>18</sup>if the discrete parameters in  $\eta$  are not chosen in such a way that  $3c - k\bar{\eta} \equiv 0(5)$  for some  $k \in \{0, 1, 2, 3, 4\}$

Let  $V_A$  and  $V_F$  be the vector spaces generated by the coefficients of the homogeneous polynomials  $A_i$  and  $F_1, G_1, H_1, F_2, G_2$ , respectively. Then (3.15) gives a (non-linear) map

$$p : V_F \rightarrow V_A \quad (3.27)$$

As we are interested actually only in the zero divisor of  $w|_{\mathcal{E}_c}$  we have to subtract in both cases above one ineffective degree of freedom.

Now the degree of the condition (3.16) is  $3e - 10r$ , thus the vanishing poses actually  $3e - 10r + 1$  conditions. So when one demands this condition of the original number  $5e - 14r + 4$  of free parameters only  $2e - 4r + 3$  remain and one is restricted to a linear subspace (or to the corresponding projective subspace)

$$U_A \subset V_A \quad (3.28)$$

Above, in the paragraph after (3.16), we investigated the question whether the concrete factorization (3.15) is even more special than what the condition (3.16) demands or whether the latter condition is also already sufficient (and thus equivalent) to imply the special form (3.15), i.e. whether the image  $\text{im } p$  of  $p$  is or is not a *proper* subset of  $U_A$

$$\text{im } p \subset U_A \quad (3.29)$$

The comparison of the number  $\dim V_F - 1$  of free parameters in the special form (3.15) with the number  $\dim U_A - 1$  of parameters left free after posing condition (3.16) gives

$$\dim V_F - 1 = 2e - 4r + 3 + (e - 5r - E) + 1 \quad (3.30)$$

$$\dim U_A - 1 = 2e - 4r + 3 \quad (3.31)$$

(note  $E \leq e - 5r$ ). The answer  $\text{im } p = U_A$  to the mentioned question (given after (3.16)) should be read combined with the concrete computations of the numbers of degrees of freedom: the *specialising subset* has codimension  $3e - 10r + 1$  in the moduli space.<sup>19</sup>

A further important issue, especially in connection with the question discussed above immediately before sect. 3.1 of whether we have to demand that  $\mathcal{C}_1 \cdot \mathcal{C}_2 = 0$  or not, is the question whether an irreducible member of the linear system  $|\mathcal{C}|$  exists at all (to see the moduli reduction effect when demanding the reducibility); the analogous condition  $\mathcal{D} \cdot \mathcal{D}' = 0$  in sect. 4 (to which we referred also in the discussion above which we just mentioned) will obstruct just this<sup>20</sup> (cf. the final paragraph of sect. 4.2).

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<sup>19</sup>The problem, alluded to earlier, remains however: whether not perhaps the divisor class of  $\mathcal{C}_i$  ( $i = 1, 2$ ) exists accidentally on  $C$  already along a *larger subset* of the moduli space.

<sup>20</sup>so there will be no question concerning the codimension of a specialising subset of the moduli space where the reducible decomposition of a certain curve exists (to pose the cohomological condition for the possibility, on a moduli subset, of an *orthogonal* decomposition is itself a choice between different components of the moduli space and not an example of moduli reduction in a given connected component)

*Remark:* The considerations which follow (not used elsewhere) in the rest of this subsection are best appreciated after having made acquaintance with the similar arguments in the final paragraph of sect. 4.2 and can be easily postponed in a first reading.

In the present section we decided not to pose this orthogonality condition of the components; but even if one would do so here the corresponding argument starting from the possible assumption  $\mathcal{C}_1 \cdot \mathcal{C}_2 = 0$  (on a subset of the moduli space where  $\mathcal{C}$  has become reducible) would not preclude the existence of an irreducible  $\mathcal{C}$ . One may contrast this with the mentioned final paragraph of sect. 4.2: there, if  $A_B = C \cap B \subset B$  degenerates to become reducible  $A_B = D + D'$ , the varying moduli in question concern the shape of  $C$ ;  $B$ , however, is not changed, and the possibility of the mentioned degeneration shows the existence of the divisor  $D$  on  $B$  (this existence as such is independent of the specific moduli chosen for  $C$ ), so it will always (independently of the moduli chosen for  $C$ ) make sense to build the intersection product  $A_B \cdot D$  in  $B$ ; from this starting point a contradiction is derived in the final paragraph of sect. 4.2 which forbids the existence of an irreducible  $A_B$  (under the assumption that an orthogonal decomposition exists).

In the example in sect. 3 the situation is different. If an irreducible  $\mathcal{C}$  exists one would have to know the following if one wants to derive a potential contradiction (which would forbid, regrettably, in the end the existence of an irreducible  $\mathcal{C}$ , given the existence of an orthogonal decomposition): one would like to argue that  $\mathcal{C} \cdot \mathcal{C}_1 \geq 0$  from the fact that this can be interpreted as a set-theoretic intersection (with no self-intersections involved). But here (different from the case on one and the same surface  $B$  on which various curves are considered from various moduli choices for  $C$ ) the components  $\mathcal{C}_i$  and a potential irreducible  $\mathcal{C}$  do not exist on one and the same surface  $C$  (for a specific moduli choice); the product  $\mathcal{C} \cdot \mathcal{C}_1$  of cohomology classes can therefore not be related to the corresponding irreducible divisors (which would imply the non-negativity; cf. also footn. 34).

This shows a further difference between potential orthogonal decompositions of the two 'standard' curves,  $\mathcal{C} = \mathcal{E}_c|_C = \mathcal{C}_1 + \mathcal{C}_2$  here and  $\sigma|_C = \mathcal{D} + \mathcal{D}'$  in sect. 4 (they arise as intersections with either the elliptic surface  $\mathcal{E}_c = \pi^{-1}(c)$  or  $B$ ): for  $B$  one gets a contradiction assuming an irreducible  $A_C = A_B$  because the curves involved ( $A_B, D, D'$ ) lie all in  $B$  and thus exist independently of the specific moduli chosen for  $C$ . So one can build  $A_B \cdot D$  as the curves exist simultaneously and derive a contradiction from that.

So, in contrast to the case in sect. 4 where we have reasons (as described in the final paragraph before sect. 3.1) to adopt the the orthogonality assumption and where it leads to dramatic restrictions (among them the nonexistence of an irreducible  $A_C$ ), in our present example it does not forbid in principle the existence of an irreducible  $\mathcal{C}$ .



### 3.3 The case $n = 4$

Finally we consider the other phenomenologically relevant case of  $n = 4$ . Here one has  $A_5 = 0 = H_1$  and thus one gets immediately also  $A_3 = 0$  (this was equ. (3.16) in the case  $n = 5$ ) as a necessary condition for the factorization

$$A_0 z^2 + A_2 x z + A_4 x^2 = (F_1 z + G_1 x)(F_2 z + G_2 x) \quad (3.32)$$

Again we ask whether this condition is also already sufficient. But the demand that the relevant number of degrees of freedom contained in the coefficients of the  $F_i, G_i$  is at least as large as the corresponding number for the  $A_i$  leads to the inequality  $2e - 4r + 4 - 1 \geq 3e - 6r + 3 - 1$  or  $\deg A_2 = e - 2r \leq 1$ , which contradicts  $\deg A_4 = e - 4r$  (for  $r = c_1 \cdot c > 0$  which itself is certainly the case if  $c_1$  is ample, say). Thus a further relation between the polynomials  $A_0, A_2, A_4$  is needed which reduces their collective number of degrees of freedom by  $e - 2r - 1$ . Of course just such a relation follows from (3.32) as one gets from the relations  $A_0 = F_1 F_2, A_2 = F_1 G_2 + G_1 F_2, A_4 = G_1 G_2$  the condition

$$A_2^2 - 4A_0 A_4 = R^2 \quad (3.33)$$

(with  $R = F_1 G_2 - G_1 F_2$ ). As the count of the reduced number of the degrees of freedom contained then in the  $A_i$  already suggests this necessary condition is now also sufficient: (3.33) gives  $A_4 | (A_2 + R)(A_2 - R)$  and one can write  $A_4 = G_1 G_2$  with  $A_2 + R = 2F_1 G_2$  and  $A_2 - R = 2G_1 F_2$  and all the polynomials  $F_i, G_i$  are now known (up to an overall constant) from the  $A_4, A_2$  and  $R$ ; taking again into account (3.33) shows that the relation  $A_0 = F_1 F_2$  is also fulfilled. Note that of the a priori possible number  $2e - 4r + 1 - 1$  of degrees of freedom on the left hand side of (3.33) only  $e - 2r + 1 = \deg R + 1$  remain.

So for  $n = 4$  one has the two equations (which are together necessary and sufficient) for the factorization:  $A_3 = 0$  and (3.33). This poses a number of  $e - 3r + 1 + e - 2r - 1$  conditions, i.e. to have the indicated decomposition of  $\mathcal{C} = \pi_C^{-1}(c)$  one restricts to a subspace of codimension  $(2\eta - 5c_1)c$ . It remains, of course, the known problem of whether a class  $\mathcal{C}_i$  of the components does not exist perhaps 'accidentally' already on a larger subset of the moduli space; on the indicated subspace the class exists naturally.

The analogue of the relation (3.17) is here (whence in particular  $\mathcal{C}_i \cdot \sigma|_C = (g_i) \cdot c$ )

$$A_B = \bar{\eta} = (a_4) = (g_1) + (g_2) \quad (3.34)$$

Furthermore one has for the divisors on  $\mathcal{E}_c$

$$\mathcal{C}_1 = \left( 2\sigma + \eta - 2c_1 - (g_2) \right) \Big|_{\mathcal{E}_c} \quad (3.35)$$

$$\mathcal{C}_2 = \left( 2\sigma + 2c_1 + (g_2) \right) \Big|_{\mathcal{E}_c} \quad (3.36)$$

Thus one gets (as intersection number in  $\mathcal{E}_c$  and also<sup>21</sup> in  $C$ )

$$\mathcal{C}_1 \cdot \mathcal{C}_2 = (2\bar{\eta} + 4c_1)c \quad (3.37)$$

Proceeding as in the case  $n = 5$  one gets with  $\pi_{C*}(\mathcal{C}_i^2) = (2c - 2\bar{\eta} - 4c_1)c$  finally ( $i = 1, 2$ )

$$c_2(V) = \eta\sigma - \frac{5}{2}c_1^2 - \frac{1}{2}\eta\bar{\eta} + 4\lambda^2 \left[ \frac{1}{2}\eta\bar{\eta} + \left( 2\bar{\eta} - 4(g_i) - 2c + 2(2\bar{\eta} + 4c_1) \right) c \right] \quad (3.38)$$

$$-N_{gen} = \lambda \left[ \eta\bar{\eta} + (2\bar{\eta} - 4(g_i))c \right] \quad (3.39)$$

Here, again, taking  $i = 1$  or  $2$  changes by (3.34) just the sign of the new term in  $N_{gen}$ .

Taking into account the parity considerations (cf. footn. 8) is much easier in our case of  $n = 4$  here than it was previously for  $n = 5$  because no parity issue on  $C$  is involved as all parity conditions are formulated on  $B$ ; furthermore the question is even completely independent of the new twist class  $\mathcal{C}_i$ . Now  $\lambda$  can be integral or strictly half-integral: in the first case one has just to demand that  $\eta \equiv c_1(2)$  on  $B$  (or equivalently  $\bar{\eta} \equiv c_1(2)$ ); for strictly half-integral  $\lambda$  one gets the condition that  $c_1$  has to be even (as  $\pi_{C*}(\mathcal{C}_i) = 2c$ ).

### 3.3.1 Some concrete examples

We take now  $n = 4$  and note that  $\eta$  and  $\eta - 4c_1$  have to be effective (classes of effective divisors), the linear system  $\eta$  has to be base point free, one has the parity condition  $\eta \equiv c_1(2)$  and  $0 \equiv c_1(2)$  for  $\lambda$  being integral and half-integral, respectively; further  $c \cong \mathbf{P}^1$  and  $\deg G_2 = E$  has to fulfil  $E \leq e - 4r = (\eta - 4c_1)c$ .

We take first, as case 1,  $B = \mathbf{P}^2$  where  $\eta = al$  (with the class  $l$  of the line  $l$ ) gives the conditions  $a \geq 12$ ,  $\lambda \in \mathbf{Z}$  and  $a$  odd. We take  $c = l$ , get the condition  $E \leq a - 12$  and

$$-N_{gen} = \lambda \left[ a(a - 12) + 2(a - 12) - 4E \right] \quad (3.40)$$

( $c_2(V)$  is computed similarly). The flexibility from  $E$  is obvious. Taking instead  $c = 2l$  one gets the condition  $E \leq 2a - 24$  and the new terms are multiplied by 2.

We take  $B = \mathbf{F}_0$  (with base  $b$  and fibre  $f$ ) as case 2 where  $\eta = xb + yf$  has to fulfil  $x, y \geq 8$  and  $x, y$  even for  $\lambda \in \mathbf{Z}$  (or no further restriction for  $\lambda \in \frac{1}{2} + \mathbf{Z}$ ). We take  $c = f$ , get the condition  $E = (g_2) \cdot c = x_g \leq x - 8$  (using the notation  $(g_2) = x_gb + y_gf$ ) and

$$-N_{gen} = \lambda \left[ x(y - 8) + y(x - 8) + 2(x - 8) - 4E \right] \quad (3.41)$$

Note that here the (easily won) examples serve just the purpose of mere illustration. By contrast in sect. 4, where we adopt the highly restrictive condition  $\mathcal{D} \cdot \mathcal{D}' = 0$  for the components of  $\sigma|_C$ , they give existence proofs for the non-emptiness of the construction.

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<sup>21</sup>assuming that  $\mathcal{C}_1$  and  $\mathcal{C}_2$  have no common component such that no self-intersection number is involved

## 4 A second example of a non-generic twist class

For our next example of a 'non-generic' class  $\chi$  we again have to go to a sublocus of the moduli space  $\mathcal{M}_V$  where a twist exists which is not available generically (but cf. the discussion in sect. 4.2). We consider the discrete parameters  $n, \eta, \lambda$  fixed and concentrate just on the connected component  $|C| = \mathbf{P}H^0(X, \mathcal{O}(C))$  of  $\mathcal{M}_V$ . This is parametrised by the possible different shapes of  $C$  lying in  $X$ ; equivalently by the possible different forms of its defining equation  $w = 0$  (up to constant rescaling) in  $X$  (variations in  $\mathcal{M}_V$  are variations in the coefficients  $a_i$  of  $w$ , up to an overall multiplicative constant).

The sublocus we are interested in is defined by assuming that the equation  $w = 0$  has a special form: we assume that the highest coefficient factorises (nontrivially: neither  $d$  nor  $d'$  is a constant)

$$a_n = d \cdot d' \quad (4.1)$$

This has the consequence that its vanishing locus  $(a_n)$ , the curve  $A_B := C \cap B \subset B$  of cohomology class  $\bar{\eta}$ , becomes reducible (where  $D = (d)$  and  $D' = (d')$  are curves<sup>22</sup> in  $B$ )

$$A_B = D + D' \quad (4.2)$$

Conversely, having such a decomposition into two curves, is equivalent<sup>23</sup> to the factorization (4.1). Among various such decompositions which exist we consider the one in (4.2), involving the curves  $D$  and  $D'$ , varying in their respective linear system. When we consider in a moment the corresponding decomposition of  $\sigma|_C$  in  $C$  we will be interested (as in the end we want to use the twist by the corresponding line bundle on  $C$ ) only in the divisor *class* of the 'new' component  $\mathcal{D}$  which occurs then on  $C$  ( $\mathcal{D}$  is just  $D$ , considered as curve in  $C$ ); therefore we take into account, already in the consideration on  $B$ , just the divisor *class*  $[D]$  of  $D$  (on  $B$  this is equivalent to fix just the cohomology class  $\delta$  of  $D$  in the corresponding cohomological decomposition  $\bar{\eta} = \delta + \delta'$ ). Furthermore, with an eye on the corresponding situation on  $C$ , we define the following two subsets of the moduli space  $\mathcal{M}_V$ : first  $\mathcal{S}_{[D]}$ , the subset where the indicated divisor class exists - but this, obviously, turns out to be the full moduli space  $\mathcal{M}_V$  (here we assume  $[D]$ , as  $[D']$ , to be just an effective divisor) - , and secondly  $\mathcal{S}_{A_B=D+D'}$ , which we define as the subspace of  $\mathcal{M}_V$  specified by the subspace of  $|(a_n)|$  where the curves of divisor class  $[A_B] = [(a_n)]$  decompose into two curves of the indicated divisor classes.

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<sup>22</sup>we will often use the same symbol for the curves and their cohomology classes; for  $(a_n) = A_B$  cf. [1]

<sup>23</sup>as  $D$  and  $D'$  represent effective divisors, such that sections of the corresponding line bundles lead back to the factors in (4.1)

The same decomposition as in (4.2) does then hold (equivalently) for the identical point set considered as curve in the surface  $C$ , i.e. for the curve  $A_C := \sigma|_C = B \cap C \subset C$  (where  $\mathcal{D}$  and  $\mathcal{D}'$  are just  $D$  and  $D'$ , but now considered as curves in  $C$ )

$$A_C = \mathcal{D} + \mathcal{D}' \quad (4.3)$$

Note that on  $B$  the curve  $D = (d)$  will of course always exist *as such* (so it is the *decomposition* (4.2) which is equivalent to the factorization (4.1)); by contrast, on  $C$  already the *existence* of the curve  $\mathcal{D}$  can not<sup>24</sup> be assumed generically; again we will be interested (for the indicated reasons of twisting) just in the existence of the divisor *class*  $[\mathcal{D}]$  of a specific curve  $\mathcal{D}$ ; this will exist (on  $C$ ) only for a subset  $\mathcal{S}_{[\mathcal{D}]}$  of  $\mathcal{M}_V$  (this subset will now be nontrivial in general, in contrast to the situation on  $B$ ). When using the twist by  $\mathcal{O}_C(\mathcal{D})$  one restricts the moduli space from  $\mathcal{M}_V$  to  $\mathcal{S}_{[\mathcal{D}]}$ . We will also consider again the subspace  $\mathcal{S}_{A_C=\mathcal{D}+\mathcal{D}'}$  where a decomposition of the concrete curve  $A_C$  into two curves with the indicated divisor classes holds on  $C$ . Obviously one has  $\mathcal{S}_{A_B=D+D'} = \mathcal{S}_{A_C=\mathcal{D}+\mathcal{D}'} \subset \mathcal{S}_{[\mathcal{D}]}$ .

To be in a region of parameters where  $C$  is non-singular one has to avoid at least obvious self-intersections. This leads one to demand

$$D \cdot D' = 0 \quad (4.4)$$

This turns out to be quite a restrictive condition; below we will treat further the question whether such an *orthogonal* decomposition (as we will call it) can be assumed to exist.

We assume that we are in the generic case where  $D$  and  $D' (\neq D)$  do not have a common component (they may be irreducible, for example); then their intersection number really counts a number of points and one has  $D \cdot D' = \mathcal{D} \cdot \mathcal{D}'$ . By contrast the self-intersection number<sup>25</sup> is sensitive to the ambient surface in which the curve is considered to lie: one has  $D^2 \neq \mathcal{D}^2$  in general (cf. the discussion at the end of sect. 4.2).

Similar remarks apply also to the curve  $A_B$  in  $B$  versus the curve  $A_C$  in  $C$ : whereas the first often is assumed to be ample (though we will not do so, cf. sect. 4.3), implying a positive self-intersection number, the latter has - again under mild assumptions, cf. below - negative self-intersection number, so it is isolated on  $C$  (by contrast the linear system  $|A_B|$  comprises, as said, easily a continuous family of equivalent divisors in  $B$ ), all this despite the fact that the same point set is concerned. Besides the sufficient difference

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<sup>24</sup>the divisor class of the pullback  $\pi_C^*D$  does not equal  $\mathcal{D}$  as, even on  $\mathcal{S}_{\mathcal{D}}$ ,  $\mathcal{D}$  will be only a component of the pullback  $\pi_C^*D$

<sup>25</sup>the self-intersection number does not just count a number of points (which can be seen just from the set-theoretic intersection, adjusted with multiplicities), but rather is the degree of the normal bundle

that this point set is, as remarked, considered as curve in different surfaces ( $B$  and  $C$ , respectively), one should also note the different meaning of the issue of 'movability' in  $B$  versus  $C$ : the movability in  $B$  means that *different* surfaces  $C$  (when  $C$  varies in its own linear system in  $X$ ) cut out different curves  $A_B = C \cap B \subset B$ ; by contrast the isolatedness of  $A_C$  refers, of course, to a *fixed* surface  $C$  (this whole discussion can be carried through analogously also for  $D \subset B$  versus  $\mathcal{D} \subset C$ ).

Furthermore, as mentioned earlier, one has<sup>22</sup>  $\pi_{C*}A_C = \bar{\eta} = A_B$ . More precisely one has even the corresponding relations for the individual components

$$\pi_{C*}\mathcal{D} = D, \quad \pi_{C*}\mathcal{D}' = D' \quad (4.5)$$

Similarly as for  $A_C$  in each case the only effect here of the projection  $\pi_C : C \rightarrow B$  is to reinterpret the relevant curve in  $C$  (which lies in the intersection  $B \cap C$ ) as a curve in  $B$ . Furthermore  $\pi_{C*}\mathcal{D}^2 = \mathcal{D}^2$ , *understood as numbers*, as will be checked below explicitly.

To compute the contributions in (2.20) and (2.21) for our case  $\chi = \mathcal{D}$  it remains to compute  $\pi_{C*}\mathcal{D}^2$  (we do not use  $D \cdot D' = 0$ ). To compute  $\mathcal{D}^2$  we compare the canonical class, considered as a number (the negative Euler number), of  $D \subset B$  and  $\mathcal{D} \subset C$

$$K_D = (K_B + D)|_D = (D - c_1)D \quad (4.6)$$

$$K_{\mathcal{D}} = (K_C + \mathcal{D})|_{\mathcal{D}} = (n\sigma|_C + \pi_C^*\eta + \mathcal{D})|_{\mathcal{D}} = (n+1)\mathcal{D}^2 + nDD' + \eta D \quad (4.7)$$

(using the cohomological relation  $D + D' = \eta - nc_1$ ) which leads to the relation of numbers

$$\mathcal{D}^2 = \frac{1}{n+1}(D - \eta - c_1 - nD')D = -c_1D - DD' \quad (4.8)$$

(here in (4.7) we made use of the relation of numbers  $\pi_C^*\eta \cdot \mathcal{D} = \pi_{C*}(\pi_C^*\eta \cdot \mathcal{D}) = \eta \cdot D$ ).

Let us check also the relation of numbers  $\pi_{C*}\mathcal{D}^2 = \mathcal{D}^2$ . Note first that  $A_C^2 = \sigma^2|_C = -c_1A_C = -c_1(\mathcal{D} + \mathcal{D}')$  (where  $c_1$  is actually  $\pi_C^*c_1$ ) is also  $\mathcal{D}^2 + \mathcal{D}'^2 + 2\mathcal{D}\mathcal{D}'$ ; furthermore one has even individually<sup>26</sup>  $\mathcal{D}^2 = -c_1\mathcal{D} - \mathcal{D}\mathcal{D}'$  and similarly for  $\mathcal{D}'$ . So one gets with the projection formula and (4.5) that  $\pi_{C*}\mathcal{D}^2 = -c_1\pi_{C*}\mathcal{D} - DD' = -c_1D - DD' = \mathcal{D}^2$ .

Thus one gets from (2.20), (2.21) in this example of  $\chi = \mathcal{D}$  (as cohomology class) the complete expressions (the first terms in the [...] brackets are the standard contributions, the terms proportional to  $D$  are the new contributions; we did not yet assume  $DD' = 0$ )

$$c_2(V) = \eta\sigma - \frac{n^3 - n}{24}c_1^2 - \frac{n}{8}\eta\bar{\eta} + \frac{n}{2}\lambda^2\eta[\bar{\eta} + 3D] \quad (4.9)$$

$$-N_{gen} = \lambda\eta[\bar{\eta} + D] \quad (4.10)$$

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<sup>26</sup>compare (4.6) and (4.7): the former gives  $K_D = (\bar{\eta} - c_1 - D')D = (\bar{\eta} - c_1 - \mathcal{D}')\mathcal{D} = (\eta - (n+1)c_1 - \mathcal{D}')\mathcal{D}$  as numbers, and the latter  $K_{\mathcal{D}} = (\eta + (n+1)\mathcal{D} + n\mathcal{D}')\mathcal{D}$  (always with suitable  $\pi_C$  pull-backs)

## 4.1 Why the component $\mathcal{D}$ of $A_C$ represents a 'new' class

Let us now investigate whether the class (of the curve)  $\mathcal{D}$  on  $C$ , which according to its definition at least looks different from the generically available classes  $\sigma|_C$  and  $\pi_C^*\phi$ , is actually 'new', i.e. not contained in the span of these 'standard' classes.

For this let us assume that one would have a relation in cohomology (where  $k \in \mathbf{Z}$ )

$$\mathcal{D} = k \sigma|_C + \pi_C^*\phi \quad (4.11)$$

The ensuing relation in  $H^2(B, \mathbf{Z})$ , which results from the projection  $\pi_{C*}$ , would then be

$$D = k A_B + n\phi \quad (4.12)$$

In other words one would get that

$$k(D + D') - D = n(-\phi) \quad (4.13)$$

Here, however, the class  $(k - 1)D + kD'$  on the left hand side will not in general be divisible by  $n$ , giving the sought-after contradiction.

Of course, it is possible that such a divisibility does hold under special circumstances, for example<sup>27</sup> when one of the classes involved is itself already divisible by  $n$ : if one would have, say,  $D = n\bar{D}$  one can just take  $k = 0$  in the resulting expression  $(k - 1)n\bar{D} + kD'$  (and analogously for  $D'$ ). In general, however, the demand of divisibility by  $n$  of the left hand side of (4.13) poses a condition which a priori need not to be fulfilled. So a relation (4.11), which would show that the class  $\mathcal{D}$  on  $C$  is not 'new', will not hold in general.

This result looks very promising in what concerns the question of moduli reduction by using a twist involving the 'new' class (or rather the corresponding line bundle). However there is still another annoying possibility which can not be excluded. Specialising the bundle moduli (actually here the equation of  $C$ ) appropriately one may be able to select a locus where  $A_B$  decomposes as described; twisting with  $\mathcal{O}(\mathcal{D})$  will then not be available generically (though the locus of availability might be larger then it seems at first sight because the class might, for some reasons, exist already on a somewhat larger subset of the moduli space). However, when posing the orthogonality condition  $D \cdot D' = 0$  for the components  $D$  and  $D'$  of  $A_B$  one has already posed a cohomological, i.e. *discrete* condition. One has not excluded the possibility that this discrete condition forces the relevant part of the moduli space  $|C|$  of deformations of  $C$  (inside  $X$ ), i.e. in our case the moduli space  $|A_B|$  of deformations of  $A_B$  (in  $B$ ) already to decompose into the corresponding moduli spaces of the deformations (in  $B$ ) of  $D$  and  $D'$ . In other words, the discrete condition might enforce already that no irreducible member of  $|A_B|$  exists.

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<sup>27</sup>more generally one has to forbid that  $D \equiv k\bar{\eta}(n)$  for any  $k = 0, \dots, n - 1$

## 4.2 The question of moduli reduction

Note first that a reducibility of one of the generically known classes<sup>28</sup> on  $C$  (in our case here the class  $\sigma = \sigma|_C = A_C$ ) does, a priori, not necessarily always introduce a 'new' class (linearly independent of the classes which are already present generically). For example, one has already generically the reducible decomposition  $\pi_C^* A_B = A_C + \tilde{A}_C$  for some further class  $\tilde{A}_C$  which is however not 'new' as it equals  $\pi_C^* A_B - \sigma|_C$ .

As already remarked above, when using the twist by  $\mathcal{O}_C(\mathcal{D})$  one restricts the moduli space from  $\mathcal{M}_V$  to  $\mathcal{S}_{[\mathcal{D}]}$  (the existence of the line bundle is equivalent to the existence of the divisor class). Now, a concrete description of the stabilized subspace  $\mathcal{S}_{[\mathcal{D}]}$  is less immediate than in the completely explicit<sup>29</sup> case of  $\mathcal{S}_{A_B=D+D'}$ : the latter is, however, in general only a subspace of the former:  $\mathcal{S}_{A_B=D+D'} \subset \mathcal{S}_{[\mathcal{D}]}$ .

To bring these two subspaces in a useful relation, i.e. to relate the stabilized subset  $\mathcal{S}_{[\mathcal{D}]}$  to the explicitly describable subset  $\mathcal{S}_{A_B=D+D'} = \mathcal{S}_{A_C=D+D'}$ , poses however the problem to go from the mere existence of the divisor *class*  $[\mathcal{D}]$  to the existence of a member of it, here a concrete *effective* divisor  $\tilde{\mathcal{D}}$ , say, which furthermore should then constitute a component of  $\sigma|_C$  (such that the curve  $\sigma|_C$  decomposes as  $\sigma|_C = \tilde{\mathcal{D}} + \tilde{\mathcal{D}}'$ , cf. (4.3)).

Note that although  $\mathcal{D}$  is by definition a component of  $\sigma|_C$  (where the latter decomposes only along a certain subset of  $\mathcal{M}_V$ ) one can not exclude the possibility that, firstly, the curve  $\mathcal{D}$  as such exists<sup>30</sup> on some surfaces  $C$  without  $\sigma|_C$  being reducible (with  $\mathcal{D}$  as a component), and, secondly, the divisor *class* of the curve  $\mathcal{D}$  may exist on an even greater subspace  $\mathcal{S}_{[\mathcal{D}]}$  of  $\mathcal{M}_V$ , i.e. the problem has actually two parts: a priori one knows only  $\mathcal{S}_{A_B=D+D'} = \mathcal{S}_{A_C=D+D'} \subset \mathcal{S}_{\mathcal{D}} \subset \mathcal{S}_{[\mathcal{D}]}$  where we denote by  $\mathcal{S}_{\mathcal{D}}$  the subspace of  $\mathcal{S}_{[\mathcal{D}]}$  where an effective member (i.e. a real curve) exists. Here the inclusions are in general not equalities and reflect the different steps of the problem referred to before; we will consider them respectively below. Both steps are not easily controlled (i.e. specialising conditions which make both inclusions equalities are not easily provided). So in this example of  $\chi = \mathcal{D}$ , where we can compute quite explicitly new contributions to the chiral matter, it is not straightforward to describe, when the twisting with  $\mathcal{O}_C(\mathcal{D})$  restricts the moduli from  $\mathcal{M}_V$  to  $\mathcal{S}_{[\mathcal{D}]}$ , how much the latter is larger than the 'known' subset  $\mathcal{S}_{A_B=D+D'}$  (which has an explicit description as a moduli space subset, cf. (4.1)).

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<sup>28</sup>i.e. the classes consisting of  $\sigma = \sigma|_C$  and the pull-back classes  $\pi_C^* \phi$  for corresponding classes  $\phi$  on  $B$

<sup>29</sup>because this subspace refers directly to a specifying condition on  $a_n$ , cf. (4.1), and the  $a_i$  directly describe the moduli space  $\mathcal{M}_V$

<sup>30</sup>this refers to a curve on each surface  $C$  corresponding to a point in a part  $\mathcal{S}_{\mathcal{D}}$  of the moduli space  $\mathbf{P}H^0(X, \mathcal{O}(C))$ , which comprises but is larger than the subset  $\mathcal{S}_{A_C=D+D'}$ , and which specialises - when going to the latter subset of the moduli space - to the component of  $\sigma|_C$  which carries the name  $\mathcal{D}$

In the *first step* (to go from  $\mathcal{S}_{[\mathcal{D}]}$  to  $\mathcal{S}_{\mathcal{D}}$ ) one has to secure the existence of an *effective* member  $\tilde{\mathcal{D}}$  in the linear equivalence class  $[\mathcal{D}]$  (i.e.  $|\mathcal{D}| \neq \emptyset$  or  $H^0(C, \mathcal{O}_C(\mathcal{D})) \neq 0$ ); although  $[\mathcal{D}]$  is by definition the *class* of an effective divisor ( $\mathcal{D}$ , which occurs as component of  $A_C$  *along a certain subset of  $\mathcal{M}_V$* ) it could happen that  $H^0(C, \mathcal{O}_C(\mathcal{D}))$  (where, despite the notation, only the divisor *class* of  $\mathcal{D}$  is actually present) is zero generically on  $\mathcal{S}_{[\mathcal{D}]}$  and jumps upwards only on a proper subset (which is  $\mathcal{S}_{\tilde{\mathcal{D}}}$  with  $\tilde{\mathcal{D}}$  an *effective* divisor in  $[\mathcal{D}]$ ).

More precisely what can be said is the following. Assume that a divisor  $\mathcal{F}$  on  $C$  exists (on  $\mathcal{S}_{[\mathcal{D}]}$ ) which, after going to  $\mathcal{S}_{\mathcal{D}}$ , becomes linearly equivalent to  $\mathcal{D}$  (which itself exists only after going to  $\mathcal{S}_{\mathcal{D}}$ ). Assume first that  $\mathcal{F}$  is *effective*: then one gets, if the linear system  $|\mathcal{F}|$  constitutes a *continuous* family, a contradiction as  $\mathcal{D}$  can not be moved in any hypothetical family of linearly equivalent divisors as it has negative self-intersection  $\mathcal{D}^2 = -c_1\mathcal{D} < 0$  when one makes the assumption - as we do from now on - that  $c_1$  is ample<sup>31</sup>; if on the other hand  $|\mathcal{F}|$  is *discrete* (an isolated  $\mathcal{F}$  may of course have, just like  $\mathcal{D}$ , a negative self-intersection number) then note that  $H^0(C, \mathcal{O}(\mathcal{F})) \cong H^0(C, \mathcal{O}(\mathcal{D}))$  is one-dimensional (as  $\mathcal{D}$  is isolated under our assumptions because of  $\mathcal{D}^2 < 0$  and as we assumed  $\mathcal{D}$  irreducible) and one has  $(s) = \mathcal{D}$  just like  $(s) = \mathcal{F}$  for a nontrivial section  $s$  of the line bundle which depends only on the divisor *class*; so such an  $\mathcal{F}$  is actually  $\mathcal{D}$  itself and one would get, for an *effective*  $\mathcal{F}$  (and under our assumptions), that  $\mathcal{S}_{[\mathcal{D}]} = \mathcal{S}_{\mathcal{D}}$ .

Now consider, however, the case that  $\mathcal{F} = \mathcal{G} - \mathcal{H}$  is a representation of a *non-effective*  $\mathcal{F}$  as difference of two effective divisors (actually one can assume that  $\mathcal{G}$  and  $\mathcal{H}$  are ample<sup>32</sup>). So, our question is, whether it is possible that  $\mathcal{G}$  becomes on  $\mathcal{S}_{\mathcal{D}}$  linearly equivalent to  $\mathcal{D} + \mathcal{H}$ ; for example, a special case would be that it becomes even equal to that combination (this is somewhat reminiscent of the decomposition  $\sigma|_C = \mathcal{D} + \mathcal{D}'$  along  $\mathcal{S}$  with the decisive difference<sup>33</sup> that  $\mathcal{D}'$  cannot be assumed to exist outside  $\mathcal{S}$ ). There are, it seems, no obvious conditions to exclude such a situation, and so this step leads to an uncontrollable modification ( $\mathcal{S}_{\mathcal{D}} \longleftrightarrow \mathcal{S}_{[\mathcal{D}]}$ ) of the relevant subset of the moduli space.

In a *second step* (to go from  $\mathcal{S}_{\mathcal{D}}$  to  $\mathcal{S}_{A_C=\mathcal{D}+\mathcal{D}'}$ ) one must ensure  $\sigma|_C$  decomposes with component  $\mathcal{D}$  from its mere existence; again there are no obvious conditions ensuring this.

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<sup>31</sup>furthermore we assume that  $\mathcal{D}$  (or  $\mathcal{D}'$ ) and a divisor representing  $c_1$  (or rather  $\pi_C^*c_1$ ) do not have a component in common: the intersection number  $c_1 \cdot \mathcal{D} := \pi_C^*c_1 \cdot \mathcal{D}$  (in  $C$ ) counts then really a (weighted) number of points and equals  $c_1 \cdot \mathcal{D} > 0$  (in  $B$ ) as  $c_1$  is assumed to be ample; we will also assume that  $\mathcal{D}$  is irreducible: this assumption implies also that a hypothetical linearly equivalent divisor  $\mathcal{F}$  cannot have a component in common with  $\mathcal{D}$  (it is also not possible that  $\mathcal{F}$  has  $\mathcal{D}$  as component); this assumption makes sure that the intersection number  $\mathcal{F}\mathcal{D}$  is really a (weighted) number of points and so non-negative

<sup>32</sup>only the divisor *class* of  $\mathcal{F}$  is important as the relevant property of  $\mathcal{F}$  is that it is linearly equivalent to  $\mathcal{D}$  (on  $\mathcal{S}_{\mathcal{D}}$ );  $\mathcal{F}$ , like any divisor, is linearly equivalent to the difference of two (very) ample divisors, cf. Ch. 1, Lemma 5, *Algebraic Surfaces and Holomorphic Vector Bundles*, R. Friedman, (1998) Springer.

<sup>33</sup>furthermore  $A_C$  can not be ample as this would give  $A_C^2 = -c_1A_C < 0$



After this general discussion let us bring in now, however, the orthogonality condition  $D \cdot D' = 0$ . This, unfortunately, excludes the existence of an irreducible member of  $|A_B|$  (so a moduli reduction effect can actually not be seen in this example; we adopt here the mild assumption  $H^2(B, \mathbf{Z})$  torsion-free)<sup>34</sup>: for in this case, with a hypothetical *irreducible* curve  $A_B$ ,  $0 \leq A_B \cdot D + A_B \cdot D' = A_B^2 \leq 0$  (as either  $A_B^2 < 0$  or  $A_B^2 \geq 0$  such that  $A_B$  is nef<sup>35</sup>, thus not<sup>37</sup> *big* (i.e.  $A_B^2 > 0$ )), such that  $D^2 = D'^2 = 0$  giving a contradiction<sup>36</sup>. This phenomenon will be substantiated in great detail in the explicit examples below.

### 4.3 Concrete examples for the decomposition

We still have to investigate how restrictive our assumption of an *orthogonal* decomposition (4.2) is (we always assume the decomposition *non-trivial*, i.e.  $D \neq 0 \neq D'$ , and *effective*, i.e.  $D$  and  $D'$  effective). Note first that in the spectral cover construction the class  $\bar{\eta}$  of  $A_B$  is assumed to be effective; often one demands further that it is even *ample* (i.e. fulfilling  $h^2 > 0$  and  $h \cdot c > 0$  for all irreducible curves  $c$ ; the individual terms  $D$  and  $D'$  can in any case not be ample because of the orthogonality). But a class  $h$  which is ample (or even only big and nef) is known<sup>37</sup> *not* to admit an orthogonal decomposition.

So, in searching for a (non-trivial, effective) orthogonal decomposition, we must assume<sup>38</sup> that the effective class  $\bar{\eta}$  is not ample (the argument for the absence of continuous moduli of the spectral line bundle on  $C$  is then not available, cf. the remarks at the end of the introduction of sect. 2), and not even big and nef. To make the discussion concrete we consider the cases  $B = \mathbf{F}_k$ , a Hirzebruch surface, or  $\mathbf{dP}_k$ , a del Pezzo surface.

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<sup>34</sup>note that we do *not* claim a contradiction from trying to argue that  $A_C \cdot \mathcal{D} = A_B \cdot D$  (as no self-intersection numbers would be concerned which for one curve can differ in different ambient surfaces) while  $A_C \cdot \mathcal{D} = \mathcal{D}^2 < 0$  (cf. (4.8), we assumed  $c_1$  to be ample) whereas  $D^2 = A_B \cdot D \geq 0$  (as  $A_B \cdot D$  is a weighted set-theoretic intersection); here (the cohomology class of) the complex curve  $\mathcal{D}$  would persist beyond  $\mathcal{S}_{\mathcal{D}}$  as (the class of) a *topological* cycle (in the family of surfaces  $C$ , parametrised by the (connected component of) the moduli space  $\mathcal{M}_V \cong \mathbf{P}H^0(X, \mathcal{O}_X(C)) \cong |C|$ ) which can coexist with (the class of) an *irreducible* member of  $|A_C|$ , thus avoiding self-intersections in  $A_C \cdot \mathcal{D}$  and making  $A_C \cdot \mathcal{D} = A_B \cdot D$  possible a priori; but we do *not* argue that these intersection products would be equal *because* they now (without self-intersections) would both equal just the weighted set-theoretic intersection: this argument is not at our disposal as  $\mathcal{D}$  persists beyond  $\mathcal{S}_{\mathcal{D}}$  only topologically but not complex analytically (by contrast on  $B$  members of  $|A_B|$  and  $|D|$  can coexist, completely independently of the moduli chosen for  $C$ , as irreducible *complex* curves thus giving  $A_B \cdot D \geq 0$ ); we do not claim  $A_C \cdot \mathcal{D} = A_B \cdot D$

<sup>35</sup>this means "numerically effective", i.e. fulfilling  $h \cdot c \geq 0$  for all irreducible curves  $c$ ; it implies  $h^2 \geq 0$

<sup>36</sup>let  $H$  be an ample divisor,  $d := HD$ ,  $d' := HD'$ ; then  $D'' := d'D - dD'$  ( $\neq 0$  adopting the technical assumption  $D \neq qD'$  for  $q$  rational) gives  $D''H = 0$  but  $D''^2 = 0$  violating the Hodge index theorem

<sup>37</sup>Ch. 1, Ex. 13, *Algebraic Surfaces and Holomorphic Vector Bundles*, R. Friedman, (1998) Springer

<sup>38</sup>this is something we have to assume for  $A_B$ ; we know already that  $A_C$  is not ample as  $A_C \cdot \mathcal{D} < 0$

### 4.3.1 Examples for $B$ a Hirzebruch surface

The surface  $\mathbf{F}_k$  is a  $\mathbf{P}^1$ -fibration over a base  $\mathbf{P}_1$  denoted by  $b$  (the fibre is denoted by  $f$ ; as no confusion arises  $b$  and  $f$  will denote also the cohomology classes). One has  $c_1(\mathbf{F}_k) = 2b + (2+k)f$  and the curve  $b$  of  $b^2 = -k$  is a section of the fibration; there is another section ("at infinity") having the cohomology class  $b_\infty = b + kf$  and the self-intersection number  $+k$ ; note that  $b_\infty \cdot b = 0$ . A class  $(x, y) := xb + yf$  is ample exactly if<sup>39</sup>  $(x, y) \cdot f > 0$  and  $(x, y) \cdot b > 0$ , i.e. if  $x > 0, y > kx$ . An irreducible non-singular curve of class  $xb + yf$  exists exactly if<sup>39</sup> the class lies in the ample cone (generated by the ample classes) or is one of the elements  $b, f$  or  $ab_\infty$  (the last only for  $k > 0$ ; here  $a > 0$ ) on the boundary of the cone; these classes together with their positive linear combinations span the effective cone  $(x, y \geq 0)$ .  $c_1$  is ample for  $\mathbf{F}_0$  and  $\mathbf{F}_1$ , whereas for  $\mathbf{F}_2$ , where  $c_1 = 2b_\infty$  (such that  $c_1 \cdot b = 0$ ), it lies on the boundary of the cone.

Let us now present certain classes  $\bar{\eta} = (x, y)$  which are effective, but *not* numerically effective, and corresponding orthogonal decompositions of the class  $\bar{\eta}$  (of  $A_B$ ): on any  $\mathbf{F}_k$  (where  $k = 0, 1, 2$ ) take  $x = 0$  and  $y \geq 2$  such that (with  $y_i > 0$  and  $y = y_1 + y_2$ )

$$(0, y) = y_1 f + y_2 f \quad (4.14)$$

and on  $\mathbf{F}_k$  with  $k = 1$  or  $2$  take  $y - kx < 0$  (and  $y > 0$ ), with  $y$  even for  $k = 2$ , such that

$$(x, y) = \left(x - \frac{y}{k}\right)b + \frac{y}{k}b_\infty \quad (4.15)$$

( $\mathbf{F}_2$  in (4.14), (4.15) is actually excluded under our assumption that  $c_1$  is ample.)<sup>40</sup>

One gets for the cohomological contributions from (4.14), say, ( $F$  the elliptic fibre)

$$c_2(V) = \eta\sigma + \left[-\frac{n^3 - n}{3} - \frac{1}{4}n^2y + n^2\lambda^2(y + 3y_1)\right]F \quad (4.16)$$

$$-N_{gen} = 2n\lambda(y + y_1) \quad (4.17)$$

(with  $\eta = (2n, 2n + nk + y)$  fulfilling the condition  $\eta \cdot b \geq 0$ ). Here the effect of turning on the new twist is seen directly in a numerical example: without the twist using  $\mathcal{D}$  (from  $D = (0, y_1)$ ) one would get here only the expressions with  $y_1 = 0$ ; this shows manifestly the greater flexibility achieved by using the twist (similarly one computes for (4.15)).

However, although the classes  $\bar{\eta}$  in (4.14), 4.15) fulfill all the postulated demands they suffer from another problem: no *irreducible* curve realising them exists<sup>41</sup>; so the reduction

<sup>39</sup>Cf. Corollary 2.18, Chap. V, *Algebraic Geometry*, R. Hartshorne, Springer Verlag (1977).

<sup>40</sup>The twist using  $\mathcal{D}$ , from  $D = (0, y_1)$  or  $(x - \frac{y}{k}, 0)$ , needs  $y_1$  or  $x - \frac{y}{k}$  even for  $n$  odd and  $k + y$  or  $x$  and  $k + y$  even for  $n$  even,  $\lambda \in \mathbf{Z}$  and  $y_1 - k$  or  $x - \frac{y}{k}$  and  $k$  even for  $n$  even,  $\lambda \in \frac{1}{2} + \mathbf{Z}$  by footn. 8.

<sup>41</sup>Actually the same is true for all of their constituents on the right hand sides of these equations except for the cases  $y = 2$  in (4.14) and  $x - y/k = 1$  in (4.15) where  $D$  and  $D'$  have irreducible representatives.

effect (from the set of *all* curves of class  $\bar{\eta}$ , including irreducible ones, to those reducible representatives corresponding to a factorisation (4.1)) can not be seen in that case<sup>42</sup>. This (and a similar result which we get below for  $B = \mathbf{dP}_k$ ) is in the end not too painful as with the choice  $\mathcal{D}$  (coming from (4.2)) of sect. 4 for the twist class  $\chi$  the moduli reduction is not under good control anyway, as described in sect. 4.2.

### 4.3.2 Examples for $B$ a del Pezzo surface

As second relevant class of base surfaces  $B$  let us consider the del Pezzo surfaces  $\mathbf{dP}_k$ : they are the blow-up of  $\mathbf{P}^2$  at  $k$  points  $P_i$  for  $k = 0, \dots, 8$  (lying suitably general, i.e. no three points lie on a line, no six on a conic); the exceptional curves from these blow-ups are denoted by  $E_i$ ,  $i = 1, \dots, k$  (one has  $\mathbf{dP}_1 \cong \mathbf{F}_1$  with  $E_1$  corresponding to  $b$ ). The intersection matrix for  $H^{1,1}(\mathbf{dP}_k)$  in the basis  $(l, E_1, \dots, E_k)$ , with  $l$  the proper transform of the line  $\tilde{l}$  from  $\mathbf{P}^2$ , is just  $\text{Diag}(1, -1, \dots, -1)$ ; furthermore  $c_1(\mathbf{dP}_k) = 3l - \sum_i E_i$  such that  $c_1^2(\mathbf{dP}_k) = 9 - k$ . On these surfaces one finds many further examples of orthogonal decompositions  $\bar{\eta} = A_B = D + D'$  (of a given curve class into classes of two curves), among them<sup>43</sup> the following families of examples on the first five del Pezzo surfaces  $\mathbf{dP}_k$ ,  $k = 1, \dots, 5$  (with<sup>44</sup>  $y = y_1 + y_2$  and  $y_i > 0$ ; further  $b, c \geq 1$  in (4.19) and the parameter  $a$  is restricted by  $1 \leq a \leq 2$  in (4.20),  $1 \leq a \leq 4$  in (4.21) and  $2 \leq a \leq 4$  in (4.22))

$$yl - yE_1 = [y_1l - y_1E_1] + [y_2l - y_2E_1] \quad (4.18)$$

$$(b + c + 1)l - (b + 1)E_1 - (c + 1)E_2 = [(b + c)l - bE_1 - cE_2] + [l - E_1 - E_2] \quad (4.19)$$

$$(a + 2)l - 2 \sum_{i=1}^2 E_i - (2a - 1)E_3 = \left[ al - \sum_{i=1}^2 E_i - (2a - 2)E_3 \right] + \left[ 2l - \sum_{i=1}^3 E_i \right] \quad (4.20)$$

$$(a + 2)l - 2 \sum_{i=1}^2 E_i - a \sum_{i=3}^4 E_i = \left[ al - \sum_{i=1}^2 E_i - (a - 1) \sum_{i=3}^4 E_i \right] + \left[ 2l - \sum_{i=1}^4 E_i \right] \quad (4.21)$$

$$(a + 2)l - 2 \sum_{i=1}^4 E_i - (2a - 3)E_5 = \left[ al - \sum_{i=1}^4 E_i - (2a - 4)E_5 \right] + \left[ 2l - \sum_{i=1}^5 E_i \right] \quad (4.22)$$

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<sup>42</sup>Actually one sees from the description given above that irreducible curve representatives exist, besides the ample classes which can not be decomposed orthogonally, only for  $b, f$  and  $ab_\infty$  for  $k > 0$  (i.e.  $k = 1$ ); but the latter is still simultaneously big and nef and the two remaining classes can obviously not be decomposed. Said differently, on  $\mathbf{F}_k$  a *decomposable class* (in our sense) has no irreducible representative (cf. for the corresponding situation on  $\mathbf{dP}_k$  the discussion around (4.25) below)

<sup>43</sup>furthermore one can easily 'enhance' a given solution: take, for example, the one in (4.18) with  $y_i = 1$ ; from this one can derive the further solution  $2l - 2E_1 - E_2 - E_3 = [l - E_1 - E_2] + [l - E_1 - E_3]$

<sup>44</sup>note that  $l \rightarrow b_\infty$  and  $E_1 \rightarrow b$  under  $\mathbf{dP}_1 \cong \mathbf{F}_1$ , so (4.18) corresponds just to (4.14) as  $l - E_1 \rightarrow f$

That all the classes here are effective<sup>45</sup> is seen when reading the class  $dl - \sum e_i E_i$  as class of the proper transform of the corresponding degree  $d$  curve in  $\mathbf{P}^2$  which goes  $e_i$  times through the points  $P_i$  and noting<sup>46</sup> that one always has

$$h_d := \frac{(d+2)(d+1)}{2} > \sum \frac{e_i(e_i+1)}{2} \quad (4.23)$$

This amounts to  $d+1 > e_1$  in the case of  $\mathbf{dP}_1$ ; here one has to distinguish the subcases  $e_1 < d$  and  $e_1 = d$ : in the last case clearly *reducible* realisations of  $\sum_{m=1}^d (l - E_1)$  exist.

That the relevant classes have *irreducible* curve representatives amounts, however, to demanding something more: the class  $2l - 2E_1$  on  $\mathbf{dP}_1$ , for example, has effective representatives but these are reducible; consider the situation on  $\mathbf{P}^2$  and fix the overall scaling of the quadratic polynomial by considering the family  $X^2 + \alpha Y^2 + \beta Z^2 + \gamma XY + \delta XZ + \epsilon YZ$ ; furthermore go to the affine ( $Z = 1$ )-patch (i.e. the  $(X, Y)$ -plane) and take  $P_1 = (0, 0)$ ; then the demand  $e_1 = 1$  for an ordinary (nonsingular) point amounts to the one condition  $\beta = 0$ , whereas a node with  $e_1 = 2$  poses the two further conditions  $\delta = \epsilon = 0$ , giving three conditions in all; the remaining two-dimensional parameter space is now, however, already exhausted by the *reducible* quadrics, i.e. the pairs of lines going through  $P_1$ , each of them with arbitrary slope (these reducible quadrics have also already a two-dimensional parameter space); more explicitly, the ensuing vanishing equation  $X^2 + \alpha Y^2 + \gamma XY = 0$  for the quadric has now the splitting form  $(X - aY)(X - bY) = 0$ . This non-existence of an *irreducible* representative is in accord with the consideration on  $\mathbf{F}_1$ : an irreducible curve representative for a class  $yf$  with  $y > 1$  does *not* exist, and these classes correspond just to the classes  $yl - yE_1$  on  $\mathbf{dP}_1$ ; rather such representatives exist, besides the classes  $b, f, ab_\infty = a(b + f)$  (with  $a > 0$ ) which correspond to  $E_1, l - E_1, al$ , just for the classes  $(x, y)$  with  $y > x > 0$  which correspond to  $yl - (y - x)E_1$ , i.e.  $dl - e_1 E_1$  with  $0 < e_1 < d$  (here the boundary cases have been discussed already: for  $e_1 = 0$ , corresponding to the cases  $db_\infty$ , irreducible representatives exist, whereas for  $e_1 = d$ , as described after (4.23), reducible realisations exist).

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<sup>45</sup>Furthermore the condition for base-point freeness of  $|\eta| = |\bar{\eta} + nc_1|$ , cf. footn. 2, is easily checked using that  $\mathbf{dP}_1 \cong \mathbf{F}_1$  and that for  $2 \leq k \leq 4$  the elements  $E_i$  and  $l - E_i - E_j$  (where  $i \neq j$ ) are generators of the effective cone of  $\mathbf{dP}_k$  with the relevant properties mentioned in footn. 2; in the example (4.22) on  $\mathbf{dP}_5$  one checks the condition also for the further generator (with the relevant properties)  $2l - \sum_{i=1}^5 E_i$ .

<sup>46</sup>Note that  $h_d = h^0(\mathbf{P}^2, \mathcal{O}(d\tilde{l}))$  so one can pose, including multiplicities,  $h_d - 1$  conditions (of going through certain points) on these degree  $d$  curves. Note also that a curve *in the original surface*  $\mathbf{P}^2$  going through a point  $P_i$  with multiplicity  $e_i$  (what poses  $\sum_{j=0}^{e_i-1} j + 1 = \sum_{j=1}^{e_i} j$  conditions, cf. Ex. 5.3, Ch. I, *Algebraic Geometry*, R. Hartshorne, Springer Verlag (1977)) would be singular for  $e_i > 1$ ; for *its proper transform in the blown-up surface*  $\mathbf{dP}_k$  however the different local branches going through  $P_i$  can be separated as they can be chosen in  $\mathbf{P}^2$  to have different slope.

This shows that the first example (4.18) on  $\mathbf{dP}_1$  can not be used for the moduli space reduction argument, in accord with the analogous comment on the corresponding case on  $\mathbf{F}_1$  at the end of sect. 4.3.1. This problem is not an accident: rather the phenomenon that here no irreducible curve realisation of the class  $A_B = \bar{\eta}$  on the left hand side of (4.2), or concretely (4.18)-(4.22), exists (and rather all realisations are exhausted by following, on the curve level, the reducible decomposition on the right hand side) holds in general. For this assume a decomposition of the mentioned class  $A_B = cl - \sum_i e_i E_i$  into the constituents  $D = al - \sum f_i E_i$  and  $D' = bl - \sum g_i E_i$  being given:

$$(a+b)l - \sum_i (f_i + g_i) E_i = \left[ al - \sum_i f_i E_i \right] + \left[ bl - \sum_i g_i E_i \right] \quad (4.24)$$

(i.e.  $c = a + b$  and  $e_i = f_i + g_i$ ). Then one computes for the difference (between the left and right hand side) of the numbers of the available degrees of freedom

$$\begin{aligned} & \frac{(a+b+1)(a+b+2)}{2} - 1 - \sum \frac{(f_i + g_i)(f_i + g_i + 1)}{2} \\ & - \left[ \frac{(a+1)(a+2)}{2} - 1 - \sum \frac{f_i(f_i + 1)}{2} + \frac{(b+1)(b+2)}{2} - 1 - \sum \frac{g_i(g_i + 1)}{2} \right] \\ & = ab - \sum f_i g_i \end{aligned} \quad (4.25)$$

The latter expression vanishes now, however, due to the orthogonality condition  $D \cdot D' = 0$  (cf. for this negative result also the corresponding situation in footn. 42 for  $\mathbf{F}_k$ ).

Let us also consider the parity issue (cf. footn. 8): using  $\mathcal{D}$ , from  $D$  in (4.18) - (4.22), needs  $y_1, b, c$  even for  $n$  odd and  $y$  odd or  $b, c$  even for  $n$  even,  $\lambda \in \mathbf{Z}$  and  $y_1$  odd for  $n$  even,  $\lambda \in \frac{1}{2} + \mathbf{Z}$ ; other cases, in particular (4.20) - (4.22), are excluded.

It is straightforward to evaluate (4.9) and (4.10) for the different cases. As this was done in sect. 4.3.1 already for (4.18) let us just consider the next infinite series in (4.19): so we take  $B = \mathbf{dP}_2$  and have

$$\eta = (b+c+1+3n)l - (b+1+n)E_1 - (c+1+n)E_2 \quad (4.26)$$

One computes as result for the chiral matter (the expression for  $c_2(V)$  is complicated)

$$-N_{gen} = \lambda \left[ (2bc + 2n(b+c) + n - 1) + (2bc + 2n(b+c)) \right] \quad (4.27)$$

To realize the enhanced flexibility of our extended ansatz using the 'new' twist class one should note the following: first of all the second large round brackets enclose the 'new' term proportional to  $D$ , cf. (4.10) (had one used  $D'$  instead, the new term would be just  $n - 1$ ); so although here the same parameters  $b$  and  $c$  occur in the standard contribution and the new contribution there is a freedom hidden here to have the second part at all.

## 5 Conclusions

An intensely studied class of supersymmetric particle physics models in four dimensions coming from string theory is that of heterotic models, built from a stable holomorphic vector bundle  $V$  on a Calabi-Yau threefold  $X$ . Two main lines of research are concerned with the particle spectrum [6], especially with respect to realistic phenomenology, and the occurring moduli [7] and their potential stabilisation. With regard to the latter the problem concerns geometric (Kähler and complex structure) moduli from  $X$  and bundle moduli. As the stabilisation of the latter is a difficult and complex task it is already interesting to restrict the bundle moduli to a smaller subspace. A possibility to achieve this is to make discrete modifications of a given bundle construction which are available only over a subset of the bundle moduli space such that the twisted bundle has less parametric freedom (i.e. turning on such discrete 'twists' constrains the moduli which thereby are restricted to a subset of their moduli space)<sup>47</sup>.

This idea can be studied concretely in the class of spectral cover bundles on elliptically fibered  $X$  [1]. At this point, remarkably, a second highly relevant issue enters the story naturally: the non-generic twists lead also to new contributions of chiral matter which modifies the standard formula [2] for the generation number  $N_{gen}$  via the appearance of new terms with new parameters. This is interesting as model builders in heterotic string theory have a long, and sometimes woebegone, experience how restrictive the simultaneous fulfillment of all the phenomenologically relevant conditions is; notable among these conditions is the one for  $N_{gen}$ . Seen from this perspective any method to gain greater flexibility in this class of models is of utmost interest. It will be even more welcomed when its use comes with the extra bonus of restricting the bundle moduli.

In the present note we develop in sect. 2 first the general form (2.20), (2.21) of the new contributions to  $c_2(V)$  and  $c_3(V)$  in the case of the spectral cover construction (this constitutes a first layer of concreteness) which are the cohomological quantities relevant for anomaly cancellation condition and the generation number, respectively. Then, in sect. 3 and 4, we compute in (3.22), (3.23), (3.38), (3.39) and (4.9), (4.10) everything explicitly in the two examples we give for the general type of the needed 'twist class' (second layer). In both cases it arises from components of a known class (of a curve on the

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<sup>47</sup>We add a word of caution to exclude possible misunderstandings: when we speak of "moving to special points in the bundle moduli space  $\mathcal{M}_V$ " to obtain new line bundles on  $C$  that can change the topology of  $V$  we understand that a corresponding twist is actually made (the topology of  $V$  as such can not change of course); thereby one reaches a *new* bundle  $V'$  which has its own moduli space  $\mathcal{M}_{V'}$  which is now the subspace of  $\mathcal{M}_V$  where the twist exists.

spectral cover surface) which becomes reducible for a special subset of the bundle moduli space (one problem occurring here is that, although the mentioned subspace where the new class occurs naturally can be given precisely, it can not be excluded that the class exists 'accidentally' already on a somewhat larger subspace)<sup>48</sup>. We also give arguments that generically the classes involved are 'new' in the sense that they do not belong to the span of the known classes. In both examples we finally specialise even further and give fully explicit examples for the two different general types of twist class (third layer): the occurrence in (3.40), (3.41) and (4.17), (4.27) of new terms with new parameters in  $N_{gen}$  shows clearly the enhanced flexibility or the more general ansatz employed. This type of procedure should be quite useful for more flexible model building.

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<sup>48</sup>another problem in the second example is that the rather strict condition adopted there that the new components are orthogonal actually forbids any irreducible representant in the first instance; this does not constitute any problem, however, for the application to the generation number

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